Multivariable Analysis

MA263 - Multivariable Analysis, Winter 2024

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Contents

1 Preface

These are lecture notes for a course given at the University of Warwick in the Winter/Spring Term 2024.

In preparation of these notes, I have freely used and copied from the excellent lecture notes from the previous course given by Mario J. Micallef, as well as the textbook by Spivak [1]. Especially, I do not claim in any way originality.

I'd be grateful for letting me know of any mistakes or typos one might find in these notes.

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2 Functions on Euclidean Space

2.1 Norm and inner product

We consider *Euclidean n-space* $\mathbb{R}^n$ as the space of all real n-tuples $(x_1, \ldots, x_n)$ with the standard addition and scalar multiplication, which makes this into a vector space. For $x \in \mathbb{R}^n$ we consider the *norm* $\|x\| := (x_1^2 + \ldots + x_n^2)^{1/2}$ and for $x, y \in \mathbb{R}^n$ the inner product $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$. Note that $\|x\| = \langle x, x\rangle^{1/2}$. We recall the following basic properties of the norm.

Proposition 2.1.1. *For $x, y \in \mathbb{R}^n$ and $a \in \mathbb{R}$ it holds*

(1) $\|x\| \geq 0$, *and* $\|x\| = 0$ *if and only if* $x = 0$.

(2) $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$; *equality holds if and only x and y are linearly dependent.*

(3) $\|x + y\| \leq \|x\| + \|y\|$.

(4) $\|ax\| = |a| \cdot \|x\|$.

The proof is left as an exercise. Note that the properties (1) and (3) together make $(\mathbb{R}^n, \|x-y\|)$ into a metric space. We summarize the properties of the scalar product in the following proposition.

Proposition 2.1.2. *For $x, x_1, x_2, y, y_1, y_2 \in \mathbb{R}^n$ and $a \in \mathbb{R}$ it holds*

(1) $\langle x, y\rangle = \langle y, x\rangle$ (symmetry).

(2) $\langle ax, y\rangle = \langle x, ay\rangle = a\langle x, y\rangle$

$\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$

$\langle x, y_1 + y_2\rangle = \langle x, y_1\rangle + \langle x, y_2\rangle$ (bilinearity).

(3) $\langle x, x\rangle \geq 0$, *and* $\langle x, x\rangle = 0$ *if and only if* $x = 0$ (positive definiteness).

(4) $\langle x, y\rangle = \frac{1}{4}\big(\|x + y\|^2 - \|x - y\|^2\big)$ (polarization identity).

Again the proof is left as an exercise.

We will denote the zero element in $\mathbb{R}^n$ by $0 = (0, \ldots, 0)$ and denote the standard basis by $e_1, \ldots, e_n$. Thus and point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ can be written as $x = \sum_{i=1}^n x_i e_i$.

2.2 The space of linear maps and matrices

Recall that a map $T : \mathbb{R}^n \to \mathbb{R}^m$ is *linear*, provided that for all $x, y \in \mathbb{R}^n$ and $a \in \mathbb{R}$ it holds

$$\begin{aligned} T(x+y) &= T(x) + T(y)\,, \\ T(ax) &= aT(x)\,. \end{aligned}$$

We denote the space of such linear maps by $L(\mathbb{R}^n, \mathbb{R}^m)$ (if $n = m$ we also write $L(\mathbb{R}^n)$). Note that $L(\mathbb{R}^n, \mathbb{R}^m)$ is itself again a (real) vectorspace.

For e_i a standard basis vector of $\mathbb{R}^n$, $i \in \{1, \ldots, n\}$, we can consider the coefficients a_{ij}, uniquely determined by

$$T(e_i) = \sum_{j=1}^{m} a_{ji} \hat{e}_j$$

where $\hat{e}_1, \ldots, \hat{e}_m$ is the standard basis of $\mathbb{R}^m$. We assign the Matrix $A = (a_{ij}) \in \mathbb{R}^{m,n}$ to T, and denote the map $T \mapsto A$ by $\mu : L(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^{m,n}$. This yields that for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $T(x) = y \in \mathbb{R}^m$ we have

$$T(x) = T\Big(\sum_{i=1}^{n} x_i e_i\Big) = \sum_{i=1}^{n} x_i T(e_i) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ji} x_i \hat{e}_j = \sum_{j=1}^{m} y_j \hat{e}_j\,.$$

Thus we have with the convention how to multiply matrices that

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Remark 2.2.1: Note that with respect to different bases $\{b_1, \cdots, b_n\}$ of $\mathbb{R}^n$ and $\{\hat{b}_1, \cdots, \hat{b}_m\}$ of $\mathbb{R}^m$ the same linear map T would have a (possibly) different matrix representation. It will be important in this course that the clearly distinguish between an (abstract) linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ and its representation as a matrix.

Given another linear map $S : \mathbb{R}^m \to \mathbb{R}^p$ with matrix representation $\mu(S) = B \in \mathbb{R}^{p,m}$ the above discussion implies that the matrix representation of the linear map $S \circ T : \mathbb{R}^n \to \mathbb{R}^p$ is given by BA, i.e. $\mu(S \circ T) = \mu(S)\mu(T) = \mathrm{BA}$.

We also recall that a norm on $L(\mathbb{R}^n, \mathbb{R}^m)$ is given by the *operator norm*

$$\|T\|_{\mathrm{op}} = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|T(x)\|}{\|x\|} = \sup_{\|x\|=1} \|T(x)\|\,.$$

Using the matrix representation $\mu(T) = A \in \mathbb{R}^{m,n}$ another norm is given by the *Frobenius norm*

$$\|A\|_F := \Big(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 \Big)^{1/2} .$$

Note that under the standard identification of $\mathbb{R}^{m,n}$ with $\mathbb{R}^{mn}$ this is just the standard norm on $\mathbb{R}^{nm}$. We have the following estimates (see Analysis III)

$$\frac{1}{\sqrt{n}} \|\mu(T)\|_F \le \|T\|_{\mathrm{op}} \le \|\mu(T)\|_F \,, \tag{2.1}$$

i.e. both norms are *equivalent*. Recall that any two norms one a finite dimensional vectorspace are equivalent, but this gives an explicit estimate.

2.3 Subsets of Euclidean Space

We recall the following basic notions.

- The open ball, centred at $x \in \mathbb{R}^n$ with radius $r > 0$ is given by
$$B(x,r) = B_r(x) = \{ y \in \mathbb{R}^n \mid \|y - x\| < r \} .$$
- A set $U \subset \mathbb{R}^n$ is called *open* if for all $x \in U$ there exists $r > 0$ s.t. $B(x,r) \subset U$.
- A set $A \subset \mathbb{R}^n$ is called *closed* if its complement $A^c := \mathbb{R}^n \setminus A$ is open.
- A set $A \subset \mathbb{R}^n$ is called *bounded* if there exists $x \in \mathbb{R}^n$ and $R > 0$ such that $A \subset B(x,R)$.
- A set $K \subset \mathbb{R}^n$ is called *compact* if it satisfies the following:
Let $\{U_\alpha\}_{\alpha \in A}$ be a family of open sets in $\mathbb{R}^n$ (indexed by a set A), such that $K \subset \cup_{\alpha \in A} U_\alpha$. Then there exists $N \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_N \in A$ such that $K \subset \cup_{i=1}^N U_{\alpha_i}$.

We recall the following equivalences.

Proposition 2.3.1. *Let $K \subset \mathbb{R}^n$. The following are equivalent*

(i) K is compact.

(ii) K is closed and bounded.

(iii) Every sequence $(x_i)_{i \in \mathbb{N}}$ in K, i.e. $x_i \in K$ for all $i \in \mathbb{N}$, contains a convergent subsequence with limit in K.

Proof. See Analysis III and Norms, Metrics and Topologies. □

2.4 Functions and continuity

For $A \subset \mathbb{R}^n$ we consider $f : A \to \mathbb{R}^m$. If $m = 1$ we call this a *scalar* function and if $m > 1$ a *vector valued* function. In case $m = n$ one also calls this a *vector field.*

For $x \in A$ we can write

$$f(x) = (f_1(x), \dots, f_m(x))$$

with $f_i : A \to \mathbb{R}$ the *component* functions for $i = 1, \dots, m$. Considering $\pi_i : \mathbb{R}^m \to \mathbb{R}, i = 1, \dots m$, the projection to the i-th coordinate (note that this is a linear function), we can write

$$f_i = \pi_i \circ f\ .$$

We call

$$graph(f) := \{(x, f(x)) \mid x \in A\} \subset \mathbb{R}^n \times \mathbb{R}^m$$

the *graph* of f. Note that *graph*(f) uniquely determines f.

Definition 2.4.1 (Continuity). *For $a \in A$ and $b \in \mathbb{R}^m$ we say*

$$\lim_{x \to a} f(x) = b$$

provided for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in A$ and $\|x - a\| < \delta$ then $\|f(x) - b\| < \varepsilon$. If $\lim_{x \to a} f(x) = f(a)$, then we say that f is continuous *at a. We say that f is continuous if it is countinous at each $a \in A$.*

As in Analysis I, it is easy to see that continuity of f at $a \in A$ is equivalent to the statement that for any sequence $(x_i)_{i \in \mathbb{N}}$ in A with $x_i \to a$ it holds that $f(x_i) \to f(a)$. But there is a further important characterisation of continuity (see Analysis III):

Theorem 2.4.2. *Let $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. Then f is continuous if and only if for every open set $U \subset \mathbb{R}^m$ there exists an open set $V \subset \mathbb{R}^n$ such that $f^{-1}(U) = V \cap A$.*

Important is also the interaction between continuous functions and compact sets (see Analysis III):

Theorem 2.4.3. *Let $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ continuous. If $K \subset A$ is compact, then $f(K) \subset \mathbb{R}^m$ is compact.*

Note that an immediate consequence of this result is that a continuous, scalar function attains its maximum and minimum on any compact set. We also recall that the composition of continuous functions is again continuous.

Theorem 2.4.4. *Let $A \subset \mathbb{R}^n, B \subset \mathbb{R}^m$ and $f : A \to \mathbb{R}^m, g : B \to \mathbb{R}^l$ be such that $f(A) \subset B$. Let $a \in A$ and assume that f is continuous at a and g continuous at $f(a)$. Then $g \circ f : A \to \mathbb{R}^l$ is continuous at a.*

Proof. This follows directly from the definition (exercise). □

Remark 2.4.5: (1) Linear maps are continuous (exercise).

(2) Let $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. Then f is continuous at $a \in A$ if and only if for all $i = 1, \ldots, m$ the component functions $f_i : A \to \mathbb{R}$ are continuous at a (exercise).

(3) All polynomials in n variables are continuous (exercise).

(4) Let $A \subset \mathbb{R}^n$ and $f, g : A \to \mathbb{R}$ be continuous. Then the quotient $x \mapsto \frac{f(x)}{g(x)}$ is continuous on $A \setminus \{x \in A \,|\, g(x) = 0\}$ (exercise).

(5) But things are not so simple. Consider

$$f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}, (x, y) \mapsto \frac{x^2 - y^2}{x^2 + y^2} .$$

Then $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist, but for every $z \in \mathbb{R}^2, z \neq 0$ the limit $\lim_{r\searrow 0} f(rz_1, rz_2)$ exists. See exercises.

2.4.1 The space $GL(n, \mathbb{R}) \subset L(\mathbb{R}^n)$ of invertible linear transformations

Recall that if a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is a bijection, then $n = m$ and $\ker T = \{0\}$. Note that the rank-nullity theorem implies that the converse is also true: a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is bijection (and thus its inverse is linear) if $m = n$ and $\ker T = \{0\}$. We can rephrase this as follows. For $A = \mu(T)$ the system $Ax = y$ of n linear equations in n variables is guaranteed a solution *for all* $y \in \mathbb{R}^n$ if and only if, should the solution exist, then it is unique, i.e. uniqueness of the solution (that is not yet known to exist!) in this setting guarantees its existence!

Definition 2.4.6. *The* general linear group over the real numbers *is given by*

$$GL(n, \mathbb{R}) := \{T \in L(\mathbb{R}^n) \,|\, T \textit{ is invertible}\},$$

with the group operation being composition of linear maps.

In terms of matrices,

$$GL(n, \mathbb{R}) := \{(a_{ij}) \in \mathbb{R}^{n,n} \mid \det(a_{ij}) \neq 0\} .$$

This is the space of nonsingular matrices with matrix multiplication as the group operation. It is easy to check that $GL(n, \mathbb{R})$ satisfies the group axioms and that it is infinite. Note that $GL(1, \mathbb{R})$ is just the set of nonzero real numbers with multiplications. We denote with $\Delta : \mathbb{R}^{n,n} \to \mathbb{R}$ the determinant function. Note that the determinant is just a polynomial of degree n on the n^2 components of an $n \times n$-matrix, and is thus continuous.

Proposition 2.4.7. *The space $GL(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n,n}$.*

Proof. We first note that $GL(n, \mathbb{R}) = \Delta^{-1}(\mathbb{R} \setminus \{0\})$. Since $\mathbb{R} \setminus \{0\}$ is open, and the determinant function is continuous, Theorem 2.4.2 yields the statement. □

The openness of $GL(n,\mathbb{R})$ means that invertibility of a linear transformation in $L(\mathbb{R}^n)$ is a stable property. In other words, an invertible linear transformation can be perturbed a little and it remains invertible. But even more is true.

Proposition 2.4.8. *The map $A \mapsto A^{-1} : GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$ is continuous.*

Proof. We recall from Linear Algebra (more precisely the Leibnitz rule for the determinant) the definition of the *adjoint matrix* $A^{\#}$, whose entries are polynomials (of degree $n-1$) in the coefficients of A. Thus $A \mapsto A^{\#} : L(\mathbb{R}^n) \to L(\mathbb{R}^n)$ is continuous. On $GL(n,\mathbb{R}) = \Delta^{-1}(\mathbb{R} \setminus \{0\})$ we furthermore can write

$$A^{-1} = \frac{1}{\det(a_{ij})} A^{\#}$$

and thus $A \mapsto A^{-1} : GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$ is continuous. □

2.4.2 Lipschitz continuity

Definition 2.4.9 (Lipschitz continuity)**.** *Let $U \subset \mathbb{R}^n$. We say that $f : U \to \mathbb{R}^k$ is* Lipschitz continuous *on U if $\exists\, M > 0$ such that*

$$\|f(x) - f(y)\| \leqslant M\|x - y\| \;\forall\, x, y \in U. \tag{2.2}$$

The Lipschitz constant M^* of f is then defined by

$$M^* := \sup_{x,y \in U,\; x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|}\,.$$

Observe that Lipschitz continuity on U implies uniform continuity on U:

$$\forall\, \varepsilon > 0, \text{if } \|x - y\| < \varepsilon/M \;\Rightarrow\; \|f(x) - f(y)\| < \varepsilon.$$

Examples of Lipschitz continuous functions

A linear map $T : \; R^n \to \mathbb{R}^m$ is Lipschitz continuous because, by linearity, $T(x) - T(y) = T(x - y)$ and therefore,

$$\|T(x) - T(y)\| = \|T(x - y)\| \leq \|T\|_{\text{op}}\|x - y\|\,,$$

i.e. the Lipschitz constant of T is equal to $\|T\|_{\text{op}}$.

By the triangle inequality, the map $x \mapsto |x| \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is Lipschitz continuous with Lipschitz constant 1:

$$\big||x| - |y|\big| \leq |x - y|.$$

Similarly, the operator norm, viewed as a function $\|\cdot\|_{\mathrm{op}} \colon L(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}$ is Lipschitz continuous:

$$\big|\|A\|_{\mathrm{op}} - \|B\|_{\mathrm{op}}\big| \leq \|A - B\|_{\mathrm{op}}.$$

The same applies to $\|\cdot\|_F$.

3 Differentiation

Motivation. Recall that $f : (b, c) \to \mathbb{R}$ is differentiable at $a \in (b, c)$ if there exists $\lambda \in \mathbb{R}$ s.t.

$$\lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f(a+h) - f(a)}{h} = \lambda\,, \tag{3.1}$$

and one denotes $f'(a) := \lambda$. Define the linear function

$$L_a : \mathbb{R} \to \mathbb{R}, x \mapsto \lambda x$$

and let

$$R(a,h) := f(a+h) - f(a) - L_a(h) \iff f(a+h) = f(a) + L_a(h) + R(a,h)\,. \tag{3.2}$$

We note that (3.1) is equivalent to

$$\lim_{\substack{h \to 0 \\ h \neq 0}} \left| \frac{R(a,h)}{h} \right| = 0\,,$$

i.e. $R(a,h)$ vanishes to higher than linear order and we can see (3.2) as stating that

$$f(a) + L_a(h) \text{ is the best affine approximation to } f(a+h) \text{ around } h = 0\,.$$

So we can restate (3.1) as:

f is differentiable at $a \in (b, c)$ if there exists a linear map $L_a : \mathbb{R} \to \mathbb{R}$ such that

$$f(a+h) = f(a) + L_a(h) + R(a,h)$$

and

$$\lim_{\substack{h \to 0 \\ h \neq 0}} \left| \frac{R(a,h)}{h} \right| = 0\,.$$

This alternative definition directly extends to higher dimensions.

3.1 The differential

Definition 3.1.1 (Differentiability)**.** *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$. We say f is differentiable at $a \in U$ if there exists a linear map $L_a : \mathbb{R}^n \to \mathbb{R}^m$ such that for all $h \in \mathbb{R}^n$ with $a + h \in U$*

it holds

$$f(a+h) = f(a) + L_a(h) + R(a,h) \tag{3.3}$$

where

$$\lim_{\substack{h\to 0\\ h\neq 0}} \frac{\|R(a,h)\|}{\|h\|} = 0\,. \tag{3.4}$$

We call L_a the derivative *of f at a, denoted by $Df(a) \in L(\mathbb{R}^n, \mathbb{R}^m)$.*

The next lemma shows that if the derivative exists, then it is unique.

Lemma 3.1.2. *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$. Assume f is differentiable at $a \in U$. Then the derivative $Df(a) : \mathbb{R}^n \to \mathbb{R}^m$ is unique.*

Proof. Assume we have for $i = 1, 2$

$$f(a+h) = f(a) + L_i(h) + R_i(a,h)$$

with

$$\lim_{\substack{h\to 0\\ h\neq 0}} \frac{\|R_i(a,h)\|}{\|h\|} = 0\,.$$

Note that

$$L_1(h) - L_2(h) = R_2(a,h) - R_1(a,h)\,,$$

and thus

$$\lim_{\substack{h\to 0\\ h\neq 0}} \frac{\|L_1(h) - L_2(h)\|}{\|h\|} = \lim_{\substack{h\to 0\\ h\neq 0}} \frac{\|R_2(a,h) - R_1(a,h)\|}{\|h\|} \leq \lim_{\substack{h\to 0\\ h\neq 0}} \frac{\|R_2(a,h)\|}{\|h\|} + \lim_{\substack{h\to 0\\ h\neq 0}} \frac{\|R_1(a,h)\|}{\|h\|} = 0\,.$$

Note that for any $x \in \mathbb{R}^n \setminus \{0\}$ and $t \searrow 0$ we have $tx \to 0$ as $t \searrow 0$. Thus

$$0 = \lim_{t\to 0} \frac{\|L_1(tx) - L_2(tx)\|}{\|tx\|} = \frac{\|L_1(x) - L_2(x)\|}{\|x\|}\,,$$

which yields $L_1(x) = L_2(x)$. Since this holds for any $x \in \mathbb{R}^n \setminus \{0\}$ we have $L_1 = L_2$. □

Definition 3.1.3 (Jacobian matrix)**.** *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be differentiable at $a \in U$. The matrix corresponding to $Df(a) \in L(\mathbb{R}^n, \mathbb{R}^m)$ (with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$) is called the* Jacobian matrix *of f at a, denoted with*

$$\partial f(a) := \mu(Df(a))\,.$$

Example 3.1.4: Let $f : \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto \sin(x)$. We claim that $Df(a,b)(x,y) = \cos(a)x =:$

$\lambda(x, y)$. Note that this is a linear map. We have

$$\lim_{\substack{(h,k)\to 0\\(h,k)\neq 0}} \frac{\|f(a+h, b+k) - f(a,b) - \lambda(x,y)\|}{\|(h,k)\|} = \lim_{\substack{(h,k)\to 0\\(h,k)\neq 0}} \frac{|\sin(a+h) - \sin a - \cos(a)h|}{\|(h,k)\|}\,.$$

Since $\sin'(a) = \cos(a)$ we have

$$\lim_{\substack{h\to 0\\h\neq 0}} \frac{|\sin(a+h) - \sin(a) - \cos(a)h|}{|h|} = 0\,.$$

Since $\|(h,k)\| \geq |h|$ this implies

$$\lim_{\substack{(h,k)\to 0\\(h,k)\neq 0}} \frac{|\sin(a+h) - \sin(a) - \cos(a)h|}{\|(h,k)\|} = 0\,,$$

and thus λ is the derivative of f at (a, b). Note that this yields that

$$\partial f(a,b) = \mu(\lambda) = (\cos(a), 0)\,.$$

As in the 1-d case, differentiability implies continuity.

Theorem 3.1.5 (Differentiability implies continuity)**.** *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be differentiable at $a \in U$. Then f is continuous at a.*

Proof. See example sheet 1. □

Assume that $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ are open and $f : U \to \mathbb{R}^m, g : V \to \mathbb{R}^l$ be such that $f(U) \subset V$. Assume that f is differentiable at $a \in U$ with differential $Df(a)$ and g differentiable at $b := f(a) \in V$ with differential $Dg(b)$. Recall that $f(a) + Df(a)(h)$ is the best affine approximation to f at $f(a)$ and $g(b) + Dg(b)(h)$ is the the best affine approximation to g at $g(a)$. So looking at $g \circ f$ a reasonable guess is that the best affine approximation of $g \circ f$ at a is $g(f(a)) + (Dg \circ Df)(a)(h)$. This is the content of the chain rule.

Theorem 3.1.6 (Chain rule)**.** *Let $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ be open and $f : U \to \mathbb{R}^m, g : V \to \mathbb{R}^l$ be such that $f(U) \subset V$. Assume that f is differentiable at $a \in U$ and g differentiable at $f(a) \in V$. Then $g \circ f : U \to \mathbb{R}^l$ is differentiable at a and*

$$D(g \circ f)(a) = Dg(f(a)) \circ Df(a)\,.$$

This yields $\partial(g \circ f)(a) = \partial g(f(a))\partial f(a)$.

Proof. Since f is differentiable at a we have $\forall h \in \mathbb{R}^n$ such that $a + h \in U$

$$f(a+h) = f(a) + Df(a)(h) + R_1(a,h) \tag{3.5}$$

with

$$\lim_{\substack{h\to 0\\ h\neq 0}} \frac{\|R_1(a,h)\|}{\|h\|} = 0\,. \tag{3.6}$$

Similarly, since g is differentiable at $b := f(a)$ we have $\forall k \in \mathbb{R}^m$ such that $b + k \in V$

$$g(b+k) = g(b) + Dg(b)(k) + R_2(b,k) \tag{3.7}$$

with

$$\lim_{\substack{k\to 0\\ k\neq 0}} \frac{\|R_2(b,k)\|}{\|k\|} = 0\,. \tag{3.8}$$

To compute the differential of $g \circ f$ at a we note that (3.5) implies that

$$b + k = f(a) + k = f(a) + Df(a)(h) + R_1(a,h)$$

and thus $k = Df(a)(h) + R_1(a,h)$. So combining (3.5) with (3.7) we get

$$\begin{aligned}(g\circ f)(a+h) &= g(f(a)) + Dg(b)\big(Df(a)(h) + R_1(a,h)\big) + R_2\big(b, Df(a)(h) + R_1(a,h)\big)\\ &= g(f(a)) + \big(Dg(f(a))\circ Df(a)\big)(h) + Dg(b)\big(R_1(a,h)\big)\\ &\quad + R_2\big(b, Df(a)(h) + R_1(a,h)\big)\,.\end{aligned}$$

To prove the statement we thus need to show that

$$\lim_{\substack{h\to 0\\ h\neq 0}} \frac{\|Dg(b)\big(R_1(a,h)\big) + R_2\big(b, Df(a)(h) + R_1(a,h)\big)\|}{\|h\|} = 0\,.$$

We can estimate

$$\lim_{\substack{h\to 0\\ h\neq 0}} \frac{\|Dg(b)\big(R_1(a,h)\big)\|}{\|h\|} \le \|Dg(b)\|_{\mathrm{op}} \lim_{\substack{h\to 0\\ h\neq 0}} \frac{\|R_1(a,h)\|}{\|h\|} = 0\,,$$

and it thus suffices to show that

$$\lim_{\substack{h\to 0\\ h\neq 0}} \frac{\|R_2\big(b, Df(a)(h) + R_1(a,h)\big)\|}{\|h\|} = 0\,.$$

To show this, choose any sequence $h_i \to 0, h_i \neq 0$ and let $k_i := Df(a)(h_i) + R_1(a,h_i)$. Note that $k_i \to 0$, and if $k_i = 0$ (since $h_i \neq 0$) we trivially have

$$\frac{\|R_2\big(b,k_i\big)\|}{\|h_i\|} = 0$$

and it is thus sufficient to assume that $k_i \neq 0$ for all i. But then we can write

$$\lim_{i\to\infty} \frac{\|R_2\big(b,k_i\big)\|}{\|h_i\|} = \lim_{i\to\infty} \frac{\|R_2\big(b,k_i\big)\|}{\|k_i\|} \frac{\|Df(a)(h_i)+R_1(a,h_i)\|}{\|h_i\|} \tag{3.9}$$

Note that by (3.6) we have

$$\frac{\|Df(a)(h_i)+R_1(a,h_i)\|}{\|h_i\|} \leq \frac{\|Df(a)(h_i)\|}{\|h_i\|} + \frac{\|R_1(a,h_i)\|}{\|h_i\|} \leq \|Df(a)\|_{\text{op}} + 1$$

for i sufficiently large. Combining this with (3.9) we have

$$\lim_{i\to\infty} \frac{\|R_2\big(b,k_i\big)\|}{\|h_i\|} \leq (\|Df(a)\|_{\text{op}}+1) \lim_{i\to\infty} \frac{\|R_2\big(b,k_i\big)\|}{\|k_i\|} = 0\,.$$

This yields the desired statement.

That $\partial(g\circ f)(a) = \partial g(f(a))\partial f(a)$ follows from $\mu(Dg(f(a))\circ Df(a)) = \mu(Dg(f(a)))\mu(Df(a))$. □

We now compute the derivative of some basic functions.

Proposition 3.1.7. *(1) Let $U\subset\mathbb{R}^n$ be open and $f: U\to\mathbb{R}^m$ be a constant function (i.e. $\exists\, y\in\mathbb{R}^m$ s.t. $f(x)=y\ \forall\, x\in U$). Then for all $a\in U$ we have $Df(a)=0$.*
(2) If $f:\mathbb{R}^n\to\mathbb{R}^m$ is linear, then for all $a\in\mathbb{R}^n$ we have $Df(a)=f$.
(3) If $s:\mathbb{R}^2\to\mathbb{R}$ is given by $s(x,y)=x+y$, then $Ds(a,b)=s$.
(4) If $p:\mathbb{R}^2\to\mathbb{R}$ is given by $p(x,y)=xy$, then $Dp(a,b)(x,y)=bx+ay$ and thus $\partial p(a,b)=(b,a)$.

Proof. (1): We clearly have $f(a+h)=f(a)$ and thus $Df(a)=0$ and $R(a,h)\equiv 0$.
(2): We have $f(a+h)=f(a)+f(h)$ and thus $Df(a)=f$ and $R(a,h)\equiv 0$.
(3): This follows from (2), since s is linear.
(4): Let $\lambda(x,y)=bx+ay$. Then

$$\begin{aligned}\lim_{\substack{(h,k)\to 0\\(h,k)\neq 0}} \frac{\|p(a+h,b+k)-p(a,b)-\lambda(h,k)\|}{\|(h,k)\|} &= \lim_{\substack{(h,k)\to 0\\(h,k)\neq 0}} \frac{|hk|}{\|(h,k)\|}\\ &\leq \lim_{\substack{(h,k)\to 0\\(h,k)\neq 0}} \frac{\frac{1}{2}h^2+\frac{1}{2}k^2}{\sqrt{h^2+k^2}} \leq \lim_{\substack{(h,k)\to 0\\(h,k)\neq 0}} \frac{\sqrt{h^2+k^2}}{2} = 0\,.\end{aligned}$$

This yields the statement. □

The following proposition relates the differentiability of a function to the differentiability of its component functions.

Proposition 3.1.8. *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$. Then f is differentiable at $a \in U$ if and only if each if its component functions $f_i, i = 1, \ldots, m$, is differentiable at a. Moreover,*

$$Df(a)(h) = \big(Df_1(a)(h), \ldots, Df_m(a)(h)\big)\,. \tag{3.10}$$

Thus $\partial f(a)$ is the $m \times n$ matrix whose i-th row *is $\partial f_i(a)$.*

Proof. "⇒": Assume f is differentiable at $a \in U$. Let $\pi_i : \mathbb{R}^m \to \mathbb{R}$ be the projection on the i-th coordinate. Note that π_i is linear, so it is differentiable and agrees with its differential at every point (Proposition 3.1.7 (2)). Thus by the chain rule $f_i = \pi_i \circ f$ is differentiable and

$$Df_i(a) = D\pi_i(f(a)) \circ Df(a) = \pi_i \circ Df(a)\,.$$

This is (3.10).

"⇐": Assume each f_i is differentiable at a with

$$f_i(a + h) = f_i(a) + Df_i(a)(h) + R_i(a, h) \tag{3.11}$$

and

$$\lim_{\substack{h\to 0\\ h\neq 0}} \frac{|R_i(a, h)|}{\|h\|} = 0\,. \tag{3.12}$$

Define

$$L(h) := \big(Df_1(a)(h), \ldots, Df_m(a)(h)\big) : \mathbb{R}^n \to \mathbb{R}^m\,.$$

Note that L is linear. We similarly define for $h \in \mathbb{R}^m$ such that $a + h \in U$

$$R(a, h) := \big(R_1(a, h), \ldots, R_m(a, h)\big)\,.$$

Then we can write (3.11) as

$$f(a + h) = f(a) + L(h) + R(a, h)\,.$$

The estimate (3.12) implies that

$$\lim_{\substack{h\to 0\\ h\neq 0}} \frac{\|R(a, h)\|}{\|h\|} = 0\,,$$

which yields that f is differentiable at a and $Df(a) = L$. (3.10) directly yields that $\partial f(a)$ is the $m \times n$ matrix whose i-th *row* is $\partial f_i(a)$. □

Corollary 3.1.9. *Let $U \subset \mathbb{R}^n$ be open and $f, g : U \to \mathbb{R}$ be differentiable at $a \in U$. Then*

(1) $D(f + g)(a) = Df(a) + Dg(a)$.

(2) $D(fg)(a) = g(a)Df(a) + f(a)Dg(a)$.

(3) Provided $g(a) \neq 0$, then

$$D\left(\frac{f}{g}\right)(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{(g(a))^2}\,.$$

Proof. We use the notation of Proposition 3.1.7.

(1): We have $f + g = s \circ (f, g)$ and so by Proposition 3.1.7 and Proposition 3.1.8

$$D(f+g)(a) = Ds(f(a), g(a)) \circ D(f,g)(a) = s \circ (Df(a), Dg(a)) = Df(a) + Dg(a)\,.$$

(2): We have $fg = p \circ (f, g)$ and so by Proposition 3.1.7 and Proposition 3.1.8

$$D(fg)(a) = Dp(f(a), g(a)) \circ D(f,g)(a) = Dp(f(a), g(a)) \circ (Df(a), Dg(a)) = g(a)Df(a) + f(a)Dg(a)\,.$$

(3): This follows from (2) since $D\left(\frac{1}{g}\right) = -\frac{1}{g^2}Dg$. □

3.2 Partial derivatives

Definition 3.2.1 (Partial derivative)**.** *Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ and $a \in U$. For $i \in \{1, \dots, n\}$ we define the i-th* partial derivative of f *at a as*

$$\partial_i f(a) := \lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f(a_1, \dots, a_i + h, \dots, a_n) - f(a)}{h}\,,$$

provided the limit exists.

Remark 3.2.2: (1) Thus $\partial_i f(a)$ is the derivative of the function

$$g(t) := f(a_1, \dots, a_{i-1}, t, a_{i+1}, \dots, a_n)$$

at $t = a_i$, i.e. $\partial_i f(a) = g'(a_i)$. This yields the interpretation that $\partial_i f(a)$ is the slope of the tangent line at $(a, f(a))$ to the curve obtained by intersection graph(f) with the plane $\{x_j = a_j \,|\, j \neq i\}$. So we know how to compute this: treat x_j for $j \neq i$ as constants and differentiate w.r.t. x_i.

(2) The partial derivative is also sometimes denoted by $\frac{\partial f}{\partial x_i}(a)$.

Definition 3.2.3 (Directional derivative)**.** *Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ and $a \in U$. For $v \in \mathbb{R}^n$ we define the* directional derivative of f at a in direction v *to be*

$$\partial_v f(a) = \lim_{\substack{t \to 0 \\ t \neq 0}} \frac{f(a + tv) - f(a)}{t}\,,$$

provided the limit exists.

Remark 3.2.4: (1) Choosing $v = e_i$ we see that $\partial_{e_i} f(a) = \partial_i f(a)$.

(2) Note that for $\lambda \neq 0$ we have

$$\begin{aligned}\partial_{\lambda v} f(a) &= \lim_{\substack{t\to 0\\ t\neq 0}} \frac{f(a+t\lambda v)-f(a)}{t} = \lambda \lim_{\substack{t\to 0\\ t\neq 0}} \frac{f(a+t\lambda v)-f(a)}{\lambda t}\\ &= \lambda \lim_{\substack{s\to 0\\ s\neq 0}} \frac{f(a+sv)-f(a)}{s} = \lambda \partial_v f(a)\,.\end{aligned}$$

(3) Assume f is differentiable at a. Note that for $I \subset \mathbb{R}$ an open interval, a function $g : I \to \mathbb{R}$ which is differentiable at $t \in I$, we have $g'(t) = Dg(t)(1)$. Consider $b : \mathbb{R} \to \mathbb{R}^n, t \mapsto a + tv$. Then we can take $g := f \circ b : I \to \mathbb{R}$, where $a \in I$. So we have

$$\partial_v f(a) = g'(0) = (f \circ b)'(0) = D(f \circ b)(0)(1) = Df(a)(Db(0)(1)) = Df(a)(v)\,.$$

Note that this directly implies that for $v, w \in \mathbb{R}^n$

$$\partial_{v+w} f(a) = Df(a)(v+w) = Df(a)(v) + Df(a)(w) = \partial_u f(a) + \partial_v f(a)\,.$$

(4) We will see in the exercises that having all directional derivatives existing at a point (which includes the partial derivatives) does not imply differentiability, not even continuity.

3.3 Relating the derivative and partial derivatives

We can now relate the partial derivatives with the Jacobi matrix.

Theorem 3.3.1 (Partial derivatives and the Jacobi matrix)**.** *Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^m$ be differentiable at $a \in U$. Then $\partial_j f_i$ exists for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. The Jacobi matrix $\partial f(a)$ is the $m \times n$ matrix $(\partial_j f_i(a))_{ij}$, i.e.*

$$\partial f(a) = \begin{pmatrix} \partial_1 f_1(a) & \cdots & \partial_n f_1(a) \\ \vdots & \ddots & \vdots \\ \partial_1 f_m(a) & \cdots & \partial_n f_m(a) \end{pmatrix}$$

Proof. Assume first that $m = 1$, i.e. $f : U \to \mathbb{R}$. Then $h : \mathbb{R} \to \mathbb{R}^n, x \mapsto (a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_n)$ is differentiable at a_j and

$$\partial_j f(a) = \partial(f \circ h)(a_j)\,.$$

Thus by the chain rule

$$\partial(f \circ h)(a_j) = \partial f(a) \partial h(a_j) = \partial f(a) e_j\,.$$

This implies that $\partial_j f(a)$ exists and is the j-th entry of the $1 \times n$ matrix $\partial f(a)$.

The theorem follows for general $m \geq 1$ since by Proposition 3.1.8 each f_i is differentiable in a and the i-th row of $\partial f(a)$ is $\partial f_i(a)$. □

Using the chain rule and that the composition of linear maps corresponds to matrix multiplication of the corresponding matrix representation yields the following corollary.

Corollary 3.3.2. *Let $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ be open and $f : U \to \mathbb{R}^m, g : V \to \mathbb{R}^l$ be such that $f(U) \subset V$. Assume that f is differentiable at $a \in U$ and g differentiable at $f(a) \in V$. Then $\partial(g \circ f)(a) = \partial g(f(a))\partial f(a)$, i.e. for $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, n\}$*

$$(\partial(g \circ f)(a))_{ij} = \sum_{k=1}^{m} \partial_k g_i(f(a))\partial_j f_k(a)\,.$$

Remark 3.3.3: For the case that $l = 1$, i.e. $g : V \to \mathbb{R}$ we see that $h = g \circ f$ satisfies for $j \in \{1, \ldots, n\}$

$$\partial_j h(a) = \sum_{k=1}^{m} \partial_k g(f(a))\partial_j f_k(a)\,.$$

We have seen in the exercises that all directional derivatives existing at a point (which includes the partial derivatives), does not imply differentiability, not even continuity. So it is clear that one needs a stronger condition to deduce differentiability from the existence of partial derivatives.

Theorem 3.3.4 (Local continuity of partial derivatives implies differentiability)**.** *Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^m$ and consider $a \in U$. Assume that all partial derivatives $\partial_j f_i$ ($i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$) exist and are continuous in an open neighborhood $V \subset U$ of a. Then f is differentiable at a.*

Proof. Due to Proposition 3.1.8, as in the proof of Theorem 3.3.1, it is sufficient to consider the case $m = 1$. Since V is open, there is $\varepsilon > 0$ such that $a + h \in V$ for all h such that $\max_i |h_i| < \varepsilon$. We can then write

$$\begin{aligned}
f(a+h) - f(a) &= f(a_1 + h_1, a_2 + h_2, \ldots, a_n + h_n) - f(a_1, a_2, \ldots, a_n) \\
&= f(a_1 + h_1, a_2 + h_2, \ldots, a_n + h_n) - f(a_1, a_2 + h_2, \ldots, a_n + h_n) \\
&\quad + f(a_1, a_2 + h_2, \ldots, a_n + h_n) - f(a_1, a_2, \ldots, a_n) \\
&= f(a_1 + h_1, a_2 + h_2, \ldots, a_n + h_n) - f(a_1, a_2 + h_2, \ldots, a_n + h_n) \\
&\quad + f(a_1, a_2 + h_2, \ldots, a_n + h_n) - f(a_1, a_2, a_3 + h_3, \ldots, a_n + h_n) \\
&\quad \vdots \\
&\quad + f(a_1, \ldots, a_{n-1}, a_n + h_n) - f(a_1, \ldots, a_{n-1}, a_n)\,.
\end{aligned} \tag{3.13}$$

Recall that $\partial_1 f(x, a_2 + h_2, \ldots, a_n + h_n)$ is the derivative of the function $x \mapsto g(x) = f(x, a_2 + h_2, \ldots, a_n + h_n)$. By assumption g is continuously differentiable on $(a_1 - \varepsilon, a_1 + \varepsilon)$. Thus by the mean value theorem applied to g we obtain

$$f(a_1+h_1,a_2+h_2,\dots,a_n+h_n)-f(a_1,a_2+h_2,\dots,a_n+h_n)=h_1\partial_1 f(\theta_1,a_2+h_2,\dots,a_n+h_n)$$

for some θ_1 between a_1 and a_1+h_1. Similarly the i-th term in the sum on the RHS of (3.13) equals

$$h_i\partial_i f(a_1,\dots,a_{i-1},\theta_i,a_{i+1}+h_{i+1},\dots,a_n+h_n)=h_i\partial_i f(c_i)$$

for some θ_i between a_i and a_i+h_i and we set $c_i:=(a_1,\dots,a_{i-1},\theta_i,a_{i+1}+h_{i+1},\dots,a_n+h_n)$. Note that $c_i\to a$ as $h\to 0$. We can then use the sum (3.13) with the above identification to estimate

$$\begin{aligned}\frac{\left|f(a+h)-f(a)-\sum_{i=1}^n\partial_i f(a)h_i\right|}{\|h\|}&=\frac{\left|\sum_{i=1}^n(\partial_i f(c_i)-\partial_i f(a))h_i\right|}{\|h\|}\\&\le\frac{1}{\|h\|}\sum_{i=1}^n|\partial_i f(c_i)-\partial_i f(a)||h_i|\\&\le\sum_{i=1}^n|\partial_i f(c_i)-\partial_i f(a)|\end{aligned}\tag{3.14}$$

Note that the final term tends to zero as $h\to 0$ since $c_i\to a$ and the partial derivatives of f are continuous on V. We can thus define the linear map (as we would expect)

$$Df(a)(h):=\sum_{i=1}^n\partial_i f(a)h_i$$

and (3.14) can be rewritten as

$$f(a+h)=f(a)+Df(a)(h)+R(a,h)$$

with

$$\lim_{\substack{h\to 0\\h\neq 0}}\frac{|R(a,h)|}{\|h\|}=0\,.$$

This shows that f is differentiable at a. □

3.3.1 The space of continuously differentiable functions

Definition 3.3.5 (Continuous differentiability)**.** *Let $U\subset\mathbb{R}^n$ be open and $f:U\to\mathbb{R}^m$ be differentiable on U. Then f is called* continuously differentiable *at $a\in U$ if the map $x\mapsto Df(x):U\to L(\mathbb{R}^n,\mathbb{R}^m)$ is continuous at a. More explicitely*

$$\forall\,\varepsilon>0,\ \exists\,\delta>0 \text{ such that if } \|x-a\|<\delta\Rightarrow\|Df(x)-Df(a)\|_{op}<\varepsilon\,.$$

Proposition 3.3.6. *Let $U\subset\mathbb{R}^n$ be open. Then $f:U\to\mathbb{R}^m$ is continuously differentiable on U if and only if $\partial f:U\to\mathbb{R}^{m,n}$ is continuous on U.*

Proof. Note that for $T \in L(\mathbb{R}^n, \mathbb{R}^m)$ we can write $\mu(T)_{ij} = \langle e_i, T(e_j)\rangle$ and thus $\mu : L(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^{m,n}$ is a continuous map.

By Theorem 3.3.1, if $Df(x)$ exists, then $(\partial_j f_i(x))_{ij} = \mu(Df(x))$. Thus if Df is continuous on U then the partial derivatives of f are continuous on U.

Conversely, if the partial derivatives of f are continuous on U then by Theorem 3.3.4 we have that $Df(x)$ exists for all $x \in U$. The equivalence of the Frobenius norm and the operator norm (see (2.1)) then directly yields that Df has to be continuous on U. □

Remark 3.3.7: Note that by this theorem, we can check continuous differentiablity by checking if the partial derivatives of f are continuous.

Notation.

$$C^1(U;\mathbb{R}^m) := \{f : U \to \mathbb{R}^m \,|\, \partial f : U \to \mathbb{R}^{m,n} \text{ is continuous}\}.$$

We also write $C^1(U) := C^1(U;\mathbb{R})$.

3.4 Geometric approximation and approximation of functions

3.4.1 Tangent to a curve

Let $\gamma\colon [a,b] \to \mathbb{R}^m$, $\gamma(t) = (x_1(t), \ldots, x_m(t))$, be a continuously differentiable parametrisation of a curve $C = \gamma([a,b]) \subset \mathbb{R}^m$. By this we mean that the functions $\frac{dx_1}{dt}, \ldots, \frac{dx_m}{dt}$ are all continuous. Assume that $\gamma'(t) = (\frac{dx_1}{dt}, \ldots, \frac{dx_m}{dt}) \neq 0 \;\forall\, t \in [a,b]$, i.e., the parametrisation γ is *regular*. Using the standard definition of derivative for all the coordinate functions of γ, we can then interpret $\gamma'(t)$ as the vector tangent to C at $\gamma(t)$.[1] The line $L_{\gamma(t)}$ tangent to C at $\gamma(t)$ is parameterised by

$$\ell(h) = \gamma(t) + \gamma'(t)h.$$

But $\gamma'(t) = \partial\gamma(t)$ and therefore, the affine linear approximation of $h \mapsto \gamma(t+h)$ by $h \mapsto \gamma(t) + \partial\gamma(t)h = \ell(h)$ is a parametrisation of the tangent line $L_{\gamma(t)}$. In other words, the affine linear approximation of $h \mapsto \gamma(t+h)$ by $h \mapsto \gamma(t) + \partial\gamma(t)h$ for small h corresponds to the geometric approximation of C by $L_{\gamma(t)}$ near $\gamma(t)$.

In the special case that C is itself a line, then $L_{\gamma(t)}$ is the same as C. This is the geometric manifestation of the fact that, the best affine linear approximation of an affine linear map is itself.

[1] We can also view $\gamma(t)$ as the position of a particle at time t and then $\gamma'(t)$, also denoted $\dot\gamma(t)$, is the velocity of the particle.

3.4.2 Tangent plane of a surface

Let $U \subset \mathbb{R}^2$ be open and let $f\colon U \to \mathbb{R}^3$ be a continuously differentiable parametrisation of a surface $S = f(U) \subset \mathbb{R}^3$. By this we mean that if $f(u,v) = (x(u,v), y(u,v), z(u,v))$ then all six partial derivatives x_u, y_u, z_u, x_v, y_v and z_v are continuous. Assume that ∂f is of rank 2, the maximal rank that it can have, at all points of U, i.e., the parametrisation f is *regular*. Since

$$f_u = (x_u, y_u, z_u),\ \ f_v = (x_v, y_v, z_v) \text{ and } \partial f = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}$$

we see that ∂f is of rank 2 if, and only if, f_u and f_v are linearly independent.[2] As in the preceding discussion for a curve C, the affine linear approximation of $(h,k) \mapsto f(u+h, v+k)$ by

$$(h,k) \mapsto f(u,v) + \partial f(u,v)(h,k) = f(u,v) + hf_u(u,v) + kf_v(u,v)$$

is then a paramtrisation of the affine plane $f(u,v) + T_{f(u,v)}S$ tangent to S at $f(u,v)$. Once again, the affine linear approximation of $(h,k) \mapsto f(u+h, v+k)$ for small h and k corresponds to the geometric approximation of S by $f(u,v) + T_{f(u,v)}S$ near $f(u,v)$.

3.4.3 Graph of a scalar function of 2 variables

Given $U \subset \mathbb{R}^2$, $g\colon U \to \mathbb{R}$, the graph of g, graph(g) is the surface parameterised by

$$f(x,y) = (x, y, g(x,y)).$$

For example, if $g(x,y) = \sqrt{1-x^2-y^2}$, $x^2+y^2 < 1$, then $f(x,y) = (x, y, \sqrt{1-x^2-y^2})$ is another parametrisation of the upper hemisphere.

Note that $f_x = (1, 0, g_x)$ and $f_y = (0, 1, g_y)$ are linearly independent for any function g. A parametrisation of the plane tangent to graph(g) at $(x, y, g(x,y))$ is given by

$$\begin{aligned}(h,k) \mapsto f(x,y) + Df(x,y)(h,k) &= (x, y, g(x,y)) + h(1,0,g_x) + k(0,1,g_y) \\ &= (x+h, y+k, g(x,y) + hg_x + kg_y) \\ &= \big(x+h, y+k, g(x,y) + \langle (h,k), \nabla g(x,y)\rangle\big).\end{aligned}$$

Thus we see that g is not differentiable at $(x_0, y_0) \in U$ if, and only if, graph(g) does not have a tangent plane at $(x_0, y_0, g(x_0, y_0))$. For example, $(x,y) \mapsto \|(x,y)\| = \sqrt{x^2+y^2}$ is not differentiable at $(0,0)$ because none of its partial derivatives exist at $(0,0)$. We see this geometrically by noting that the graph of $(x,y) \mapsto \|(x,y)\|$ on $\mathbb{R}^2$ is a rotationally symmetric cone about the z-axis with an apex at the origin where the cone does not have a tangent plane.

---

[2] For example,

$$f(u,v) := ((\cos v)(\sin u), (\sin v)(\sin u), \cos u), \quad 0 < v < 2\pi,\ 0 < u < \pi,$$

is a regular parametrisation of the unit sphere minus the prime meridian, i.e., the semicircle running from the North Pole $(0,0,1)$ to the South Pole $(0,0,-1)$ via $(1,0,0)$.

Orders of approximation of a function

For arbitrary values of n and m, it is not possible to provide simple geometric interpretations of $Df(x) \in L(\mathbb{R}^n, \mathbb{R}^m)$ similar to those presented above; consider, for example, $n = 3$ and $m = 2$. Therefore we have to change our viewpoint when defining the derivative from that of rate of change or tangent line and tangent plane to that of best approximation by a linear map. Linear maps are the simplest maps, after constant maps, and they are fully understood (rank, eigenvalues, etc.) by the methods of linear algebra. We can then transfer this knowledge of linear maps to differentiable maps up to an error that can be quantified by (3.4).

Recalling Taylor's theorem, we see that

(i) a function $h \mapsto f(x+h)$ which is continuous at $h = 0$ admits an approximation by the constant $f(x)$. The error of the approximation is measured by $\varepsilon = \varepsilon|h|^0$ and therefore, this approximation is said to be of zeroth order in h.

(ii) a function $h \mapsto f(x+h)$ which is differentiable at $h = 0$ can be approximated by the affine linear map $h \mapsto f(x) + Df(x)\,h$. According to (3.4), the error of the approximation is now measured by $\varepsilon|h|$ and therefore, this approximation is said to be of first order (equivalently, linear) in h. Furthermore, for small h, $\varepsilon|h| \ll \varepsilon$, i.e., this first order approximation is much better (i.e., the error is smaller) than that demanded by continuity, or even Lipschitz continuity.

(iii) Later on in this module, we shall show that if $h \mapsto f(x+h)$ is twice differentiable at $h = 0$ then it admits an approximation of the form $h \mapsto f(x) + Df(x)\,h +$ (*quadratic polynomial* in h). The error of the approximation is now measured by $\varepsilon|h|^2$ and therefore, this approximation is said to be of second order (equivalently, quadratic) in h. Quadratic polynomials are also studied in linear algebra under the topic of symmetric bilinear forms.

The above discussion should make clear that, when discussing derivatives of functions of several variables, the significance of derivative moves away from that of rate of change to that of approximation by polynomials which are 'simple' enough to be amenable to detailed study.

4 The inverse function theorem

Motivation. Consider $I \subset \mathbb{R}$ an open interval and $f : I \to \mathbb{R}$ be *continuously* differentiable on I. Assume for $a \in I$ that $f'(a) > 0$ (alternatively $f'(a) < 0$). Since $x \mapsto f'(x)$ is continuous, there exists an open subinterval $J \subset I, a \in J$ such that $f'(x) > 0$ (alternatively $f'(x) < 0$) for all $x \in J$. Thus the mean value theorem yields that f is strictly increasing (strictly decreasing) on J. This implies that f is injective on J and $W = f(J) \subset \mathbb{R}$ is an open interval. From Analysis I we know that the inverse $f^{-1} : W \to J$ exists and is differentiable on W with

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

Recall that this formula follows directly from differentiating the relation $y = f(f^{-1}(y))$ in y and using the chain rule.

Remark 4.0.1: (1) Note that we do not have a control on the size of J.

(2) Note that $f'(a) \neq 0$ implies that the derivative of f at a, seen as a linear map

$$Df(a) : \mathbb{R} \to \mathbb{R}, h \mapsto f'(a)h$$

is invertible.

(3) We can alternatively look at $\text{graph}(f) \subset \mathbb{R}^2$. From the discussion in Section 3.4 we recall that differentiability of f at a is equivalent to the statement that $\text{graph}(f)$ is well approximated around $(a, f(a))$ by the affine line

$$L := \{(a + h, f(a) + f'(a)h) \in \mathbb{R}^2 \mid h \in \mathbb{R}\}.$$

Note that $f'(a) \neq 0$ is equivalent to the statement that L is *not* parallel to the x-axis, i.e. both the projections π_x, π_y to each coordinate axis are bijections between L and the x-axis as well as L and the y-axis, respectively. The continuous differentiability of f then yields that the same holds for $\text{graph}(f)$ in a neighboorhood of $(a, f(a))$. This is equivalent to the local invertibility of f.

Recall that in higher dimension a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ is invertible if and only if $m = n$ and $\det L \neq 0$. So if we want to show local invertability of a map $f : U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^m$ is open, we should assume $m = n$ and $\det Df(a) \neq 0$ as well as that f is continuously differentiable. This will be the content of the inverse function theorem. We first need two little lemmata.

Lemma 4.0.2. *Let $U \subset \mathbb{R}^n$ be open. Assume that $f : U \to \mathbb{R}$ has a local minimum (maximum) at*

$a \in U$, that is there exists $\varepsilon > 0$ such that

$$f(x) \geq f(a) \quad \big(f(x) \leq f(a)\big) \quad \forall\, x \in B(a,\varepsilon) \subset U \ .$$

Assume that for some $i \in \{1,\dots,n\}$, $\partial_i f(a)$ exists. Then $\partial_i f(a) = 0$.

Proof. See Question 4 on Example sheet 2. □

Lemma 4.0.3. *Let $B(a,r) \subset \mathbb{R}^n$ and $f : B(a,r) \to \mathbb{R}^n$ be such that all partial derivatives $\partial_j f_i$ exists on $B(a,r)$ and*

$$|\partial_j f_i(x)| \leq M$$

for all $x \in B(a,r)$, $i,j \in \{1,\dots,n\}$. Then

$$\|f(x) - f(y)\| \leq n^2 M \|x - y\|$$

for all $x, y \in B(a,r)$.

Proof. See the proof of Question 5 on Example Sheet 2. □

Theorem 4.0.4 (Inverse Function Theorem)**.** *Let $U \subset \mathbb{R}^n$ be open and assume $f : U \to \mathbb{R}^n$ is continuously differentiable on U. For $a \in U$ assume further that $\det(Df(a)) \neq 0$ (alternatively $\det(\partial f(a)) \neq 0$). Then there is an open neighborhood $V \subset U$ of a and an open neighborhood $W \subset \mathbb{R}^n$ of $f(a)$ such that $f : V \to W$ has a continuous inverse $f^{-1} : W \to V$ which is continuously differentiable and*

$$D(f^{-1})(y) = \big(Df(f^{-1}(y))\big)^{-1} \tag{4.1}$$

for all $y \in W$.

Proof. Define $\lambda := Df(a) \in L(\mathbb{R}^n)$. Since $\det(Df(a)) = \det(\lambda) \neq 0$ we see that λ is invertible. Note that the Inverse Function Theorem holding for f is equivalent to it holding for $\lambda^{-1} \circ f$. But we have by the chain rule

$$D(\lambda^{-1} \circ f)(a) = D(\lambda^{-1})(f(a)) \circ Df(a) = \lambda^{-1} \circ Df(a) = I_n \, ,$$

where I_n is the identity map on $\mathbb{R}^n$. Thus we can w.l.og. assume that $Df(a) = \lambda = I_n$.

Step 1: *f is injective in a neighborhood around a.*

Since f is continuously differentiable on U we can choose $\varepsilon > 0$ (sufficiently small) such that on $B(a,\varepsilon) \subset U$ it holds

$$\det(Df(x)) \neq 0 \qquad \forall\, x \in B(a,\varepsilon) \tag{4.2}$$

$$\big|\partial_j f_i(x) - \delta_{ji}\big| = \big|\partial_j f_i(x) - \partial_j f_i(a)\big| < \frac{1}{2n^2} \quad \forall\, 1 \leq i,j \leq n \text{ and } x \in B(a,\varepsilon) \tag{4.3}$$

We then consider $g(x) = f(x) - x$ on $B(a,\varepsilon)$. By (4.3)

$$\left|\partial_j g_i(x)\right| < \frac{1}{2n^2} \quad \forall\, 1 \le i,j \le n \text{ and } x \in B(a,\varepsilon)\,.$$

Thus Lemma 4.0.3 yields

$$\|f(x) - x - (f(y) - y)\| = \|g(x) - g(y)\| \le \frac{1}{2}\|x - y\| \qquad \forall\, x,y \in B(a,\varepsilon)\,. \tag{4.4}$$

Note that instead of $g(x) = f(x) - x$ we could have chosen

$$\tilde{g}(x) = f(x) - \big(f(a) + Df(a)(x-a)\big) = f(x) - x - f(a) + a$$

to arrive at the same result. I.e. with (4.4) we are measuring here how quickly f deviates from its affine linear approximation at a. Combining (4.4) with the reverse triangle inequality yields

$$\|x - y\| - \|f(x) - f(y)\| \le \|f(x) - x - (f(y) - y)\| \le \frac{1}{2}\|x - y\|$$

and thus

$$\|x - y\| \le 2\|f(x) - f(y)\| \qquad \forall\, x,y \in B(a,\varepsilon)\,, \tag{4.5}$$

which yields the desired injecitvity. Note that by continuity (4.5) extends to all $x,y \in \overline{B(a,\varepsilon)}$.

Step 2: *f is surjective in a neighborhood around a.*[1]

Note that (4.5) implies that for all $x \in \partial B(a,\varepsilon)$

$$\|f(x) - f(a)\| \ge \frac{\varepsilon}{2} =: d\,.$$

Let $W := \{w \in \mathbb{R}^n \mid \|w - f(a)\| < d/2\} = B(f(a), d/2)$. Thus if $w \in W$ and $x \in \partial B(a,\varepsilon)$ we have

$$\|w - f(a)\| < \frac{d}{2} \le \|w - f(x)\|\,, \tag{4.6}$$

since $f(x) \notin B(f(a), d)$.

Claim: *For any $w \in W$ there exists a unique $x \in B(a,\varepsilon)$ such that $f(x) = w$.*

Consider

$$g : \overline{B(a,\varepsilon)} \to \mathbb{R},\ g(x) := \|w - f(x)\|^2 = \sum_{j=1}^{n} (w_j - f_j(x))^2\,.$$

Note that g is clearly continuous and thus attains its minimum on $\overline{B(a,\varepsilon)}$. If $x \in \partial B(a,\varepsilon)$, by (4.6), we have

$$g(a) < g(x)$$

and thus the minimum is not attained on $\partial B(a,\varepsilon)$. But g is clearly differentiable on $B(a,\varepsilon)$, so

[1] See Section A.1 in the appendix for a proof using the contraction mapping principle.

Lemma 4.0.2 yields that there is a point $x \in B(a,\varepsilon)$ (where g attains its minimum) such that $\partial_i g(x) = 0$ for all $i = 1, \dots, n$, i.e.

$$2\sum_{j=1}^{n}(w_j - f_j(x))\partial_i f_j(x) = 0\,.$$

Denoting the column vector (i.e. the $n \times 1$ matrix) with entries $w_j - f_j(x)$ by p we see that we can write this equality as

$$2(\partial f(x))^T p = 0\,.$$

But since by (4.2) $\det((\partial f(x))^T) = \det(\partial f(x)) \neq 0$, this implies $p = 0$ and thus $w = f(x)$. Uniqueness follows from (4.5).

Step 3: *Continuity and differentiability of* f^{-1}.

Let $V := B(a,\varepsilon) \cap f^{-1}(W)$ (which is open, since f is continuous). We have shown that $f : V \to W$ has an inverse $f^{-1} : W \to V$. We can rewrite (4.5) as

$$\|f^{-1}(v) - f^{-1}(w)\| \leq 2\|v - w\| \qquad \forall\, v, w \in W\,, \tag{4.7}$$

and thus f^{-1} is (Lipschitz) continuous.

It remains to show that f^{-1} is differentiable. Let $x \in V$ and $\mu := Df(x) \in GL(\mathbb{R}^n)$

Claim: f^{-1} is differentiable at $w = f(x)$ and $D(f^{-1})(w) = \mu^{-1}$.

We first note that the claim that $D(f^{-1})(w) = \mu^{-1} = (Df(x))^{-1}$ gives (4.1). Note that since f is continuously differentiable, (4.1) implies that f^{-1} is continuously differentiable (recall Proposition 2.4.8). Furthermore, by definition

$$f(y) = .f(x) + \mu(y - x) + R(x, y - x)$$

where

$$\lim_{\substack{y \to x \\ y \neq x}} \frac{\|R(x, y - x)\|}{\|y - x\|} = 0\,, \tag{4.8}$$

and thus

$$\mu^{-1}(f(y) - f(x)) = y - x + \mu^{-1}(R(x, y - x))\,.$$

Since $x = f^{-1}(w)$ and there is a unique $v \in W$ such that $y = f^{-1}(v)$, we can write this as

$$f^{-1}(v) = f^{-1}(w) + \mu^{-1}(v - w) - \mu^{-1}\big(R(f^{-1}(w), f^{-1}(v) - f^{-1}(w))\big)$$

So we need to show that

$$\lim_{\substack{v \to w \\ v \neq w}} \frac{\|\mu^{-1}R(f^{-1}(w), f^{-1}(v) - f^{-1}(w)\|}{\|v - w\|} = 0\,. \tag{4.9}$$

Note that for $v \neq w$ we can estimate

$$\begin{aligned}(4.10)\qquad \frac{\|\mu^{-1}R(f^{-1}(w), f^{-1}(v) - f^{-1}(w)\|}{\|v-w\|} &\leq \|\mu^{-1}\|_{\mathrm{op}} \frac{\|R(f^{-1}(w), f^{-1}(v) - f^{-1}(w))\|}{\|f^{-1}(v) - f^{-1}(w)\|} \\ &\quad \cdot \frac{\|f^{-1}(v) - f^{-1}(w)\|}{\|v-w\|}\end{aligned}$$

Note that since f^{-1} is continuous we have $f^{-1}(v) \to f^{-1}(w)$ as $v \to w$ and so (4.8) yields

$$\lim_{\substack{v \to w \\ v \neq w}} \frac{\|R(f^{-1}(w), f^{-1}(v) - f^{-1}(w))\|}{\|f^{-1}(v) - f^{-1}(w)\|} = 0\,.$$

But (4.7) implies that the second term on the RHS in (4.10) is bounded from above by 2, so we obtain (4.9). □

Example 4.0.5: Consider $f : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (xy, x^2 + y^2)$. We consider (z, w) in the image, i.e. $z = f_1(x, y) = xy$ and $w = f_2(x, y) = x^2 + y^2$. We first note that $f(\mathbb{R}^2) \subset \Omega := \{(z, w) \in \mathbb{R}^2 \,|\, w \geq 0\}$. Note that we have

$$\partial f(x, y) = \begin{pmatrix} y & x \\ 2x & 2y \end{pmatrix}$$

and thus $\det(\partial f(x, y)) = 2y^2 - 2x^2$. Thus $Df(x, y)$ is invertible for $x \neq \pm y$.

For $(z, w) \in \Omega$ we want to solve $(x, y) = f^{-1}(z, w)$. For $z \neq 0$ we have $y = z/x$ and thus

$$w = x^2 + y^2 = x^2 + \frac{z^2}{x^2} \;\Leftrightarrow\; x^4 - x^2 w + z^2 = 0\,,$$

which has solutions

$$x = \pm\left(\frac{w \pm \sqrt{w^2 - 4z^2}}{2}\right)^{1/2} \qquad y = \frac{z}{x} = \pm z\left(\frac{w \pm \sqrt{w^2 - 4z^2}}{2}\right)^{-1/2}$$

provided $0 \leq w^2 - 4z^2 = (x^2 + y^2)^2 - 4x^2y^2 = (x^2 - y^2)^2$. Note that the choice of signs for x determines y. I.e. we have for $z \neq 0$:

- If $w^2 - 4z^2 > 0$ then any point (z, w) has 4 preimages and the above formulas yield 4 (smooth) local inverses for f. Note that in this case we are in a stable situation: changing (z, w) a bit still yields 4 solutions.

- If $w^2 - 4z^2 = 0$, (i.e. $x = \pm y$) then any point (z, w) has 2 preimages. Note that in this case we are in an instable situation: changing (z, w) a bit still yields either 4 or 2 or no solutions.

- If $w^2 - 4z^2 < 0$, then $(z, w) \notin f(\mathbb{R}^2)$. Again this is a stable situation: changing (z, w) a bit still yields no solution.

For $z = 0$ we have either $x = 0$ and $y = \pm\sqrt{w}$ or $y = 0$ and $x = \pm\sqrt{w}$. So in this case we also have

4 preimages provided $w \neq 0$ (this is still the stable case above). For $(z, w) = (0, 0)$ we only have one preimage $(x, y) = (0, 0)$

Computing $\partial f^{-1}(z, w)$ is difficult. But $\partial f^{-1}(z, w) = (\partial f(x, y))^{-1}$ and thus

$$\partial f^{-1}(z, w) = \begin{pmatrix} y & x \\ 2x & 2y \end{pmatrix}^{-1} = \frac{1}{2(y^2 - x^2)} \begin{pmatrix} 2y & -x \\ -2x & y \end{pmatrix}$$

Geometric interpretation. Note that by fixing $z \neq 0$ we have $xy = z$, i.e. (x, y) lie on the hyperbolas determined by $xy = z$. Fixing $w > 0$ gives that $w = x^2 + y^2$, i.e. (x, y) lies on a circle with radius $w^{1/2}$. So if $w^2 > 4z^2$, then the circe of radius $w^{1/2}$ intersects the hyperbolas $\{(x, y) \in \mathbb{R}^2 \,|\, xy = z\}$ in 4 points. If $w^2 < 4z^2$ then the circle of radius $w^{1/2}$ does not intersect the hyperbolas $\{(x, y) \in \mathbb{R}^2 \,|\, xy = z\}$. If $w^2 = 4z^2$ then the circle of radius $w^{1/2}$ touches the hyperbolas $\{(x, y) \in \mathbb{R}^2 \,|\, xy = z\}$ in 2 points.

Note that the case $w^2 > 4z^2$ is the stable case, i.e. jiggling (z, w) a bit gives always 4 solutions. But the last case (i.e. $w^2 = 4z^2$, that is $x = \pm y$) is the instable case (i.e. where $\det(Df) = 0$). Jiggling (z, w) a bit gives either 4 solutions, 2 solutions or no solution!

5 The implicit function theorem

Motivation. Consider $f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = x^2 + y^2 - 1$. Note that the set $S := \{(x, y) \in \mathbb{R}^2 \,|\, f(x, y) = 0\}$ is the unit circle. Thus if $f(a, b) = 0$ and $a \neq \pm 1$ there exist open intervals $I \ni a, J \ni b$ such that for all $x \in I$ there exists a unique $y \in J$ (i.e. $(x, y) \in I \times J$) such that $f(x, y) = 0$.

So we can define $g : I \to J$ by $g(x) = y$ and thus for all $x \in I$ we have $f(x, g(x)) = 0$. Explicitly in our case, we have

$$g_+(x) = \sqrt{1 - x^2} \;\text{ if } b > 0 \quad \text{and} \quad g_-(x) = -\sqrt{1 - x^2} \;\text{ if } b < 0\,.$$

Note that $g_\pm$ are differentiable if $x \neq \pm 1$.

We say that $g_\pm$ are implicitly defined by the equation $f(x, y) = 0$. Note that if $a = \pm 1$ is impossible to find such a function g. Nevertheless, note that around $(\pm 1, 0)$ we can write x as a (smooth) function of y.

Differentiating the equation $f(x, g(x)) = 0$ yields by the chain rule

$$\partial_x(f(x, g(x))) = 0 \;\Leftrightarrow\; \partial_x f(x, g(x)) + \partial_y f(x, g(x))\partial_x g(x) = 0$$

and thus

$$\partial_x g(x) = -(\partial_y f(x, g(x)))^{-1}\partial_x f(x, g(x))\,,$$

provided $\partial_y f(x, g(x)) \neq 0$. In our case at hand this yields

$$\partial_x g_\pm(x) = -\frac{x}{y} = -\frac{x}{g_\pm(x)} = \mp\frac{x}{\sqrt{1 - x^2}}\,,$$

which we can check by explicitly differentiating $g_\pm$.

A more complicated example. Take $f : \mathbb{R}^3 \to \mathbb{R}, f(x, y, z) = y^2 + xz + z^2 - e^z$ and consider the set $S := \{(x, y, z) \in \mathbb{R}^3 \,|\, f(x, y, z) = 0\}$. Note that we can't explicitly solve for z in terms of x, y. Nevertheless, assume that in a neighborhood of $(a, b, c) \in S$ there is a smooth function $g(x, y)$ such that $z = g(x, y)$ for all $(x, y, z) \in S$. Then we can compute the partial derivatives of g (without knowing g):

$$\begin{aligned} 0 &= \partial_x(f(x, y, g(x, y)) = \partial_x f(x, y, g(x, y)) + \partial_z f(x, y, g(x, y))\partial_x g(x, y)\,, \\ 0 &= \partial_y(f(x, y, g(x, y)) = \partial_y f(x, y, g(x, y)) + \partial_z f(x, y, g(x, y))\partial_y g(x, y)\,, \end{aligned}$$

and thus at $z = g(x, y)$

$$\partial_x g(x,y) = -\frac{\partial_x f(x,y,z)}{\partial_z f(x,y,z)} = -\frac{z}{x+2z-e^z}$$
$$\partial_y g(x,y) = -\frac{\partial_y f(x,y,z)}{\partial_z f(x,y,z)} = -\frac{2y}{x+2z-e^z}$$

So at $(-2, e, 2) \in S$ we have

$$\partial_x g(-2,e) = -\frac{2}{2-e^2}$$
$$\partial_y g(-2,e) = -\frac{2e}{2-e^2}\,.$$

General situation. Consider $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ and for a point $z \in \mathbb{R}^n \times \mathbb{R}^m$ write $z = (x, y)$ with $x \in \mathbb{R}^n, y \in \mathbb{R}^m$. Assume we have functions $f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ for $i = 1, \ldots, m$ (i.e. $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$). Assume $f_i(a, b) = 0$ for $i = 1, \ldots, m$. When can we find for each $x \in \mathbb{R}^n$ near a a unique point $y \in \mathbb{R}^m$ near b such that $f_i(x, y) = 0$ for $i = 1, \ldots, m$?

Let us consider the linear case, i.e. $L \in L(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^m)$. This uniquenely determines $S \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $T \in L(\mathbb{R}^m, \mathbb{R}^m)$ such that $L(z) = L(x, y) = S(x) + T(y)$ (i.e. $S(x) := L(x, 0)$ and $T(y) := L(0, y)$). We then want to solve

$$0 = L(z) = S(x) + T(y)$$

uniquely for $y \in \mathbb{R}^m$. This implies that $T \in L(\mathbb{R}^m, \mathbb{R}^m)$ has to be invertible and

$$y = -T^{-1}(S(x)) =: g(x),$$

i.e. we can write y as a function of x.

Motivated by the linear case (and having the Inverse Function Theorem in mind), we can formulate the implicit function theorem.

Theorem 5.0.1 (Implicit Function Theorem)**.** *Let $U \subset \mathbb{R}^n \times \mathbb{R}^m$ be open and $f : U \to \mathbb{R}^m$ be continuously differentiable. For $(a, b) \in U$ such that $f(a, b) = 0$ assume that the $m \times m$ matrix $M = (\partial_{n+j} f_i(a, b))_{i,j}$ is invertible, i.e.* $\det(M) \neq 0$*. Then there is $A \subset \mathbb{R}^n$ an open neighborhood of a and $B \subset \mathbb{R}^m$ an open neighborhood of b and a continuously differentiable function $g : A \to B$ such that $f(x, g(x)) = 0$ for all $x \in A$.*

Proof. We aim to apply the inverse function theorem. Note to do that we need a function from $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ whose derivative is invertible at (a, b). Define

$$F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m, F(x, y) = (x, f(x, y))$$

Note that we then have schematically

$$\partial F(a,b) = \begin{pmatrix} I_n & 0 \\ * & M \end{pmatrix}$$

and thus $\det(\partial F(a,b)) = \det(M) \neq 0$. By the Inverse Function Theorem, Theorem 4.0.4 there is an open neighborhood $V \subset \mathbb{R}^n \times \mathbb{R}^m$ of (a,b), which we can assume is of the form $V = A \times B$ ($A \subset \mathbb{R}^n$ open and $B \subset \mathbb{R}^m$ open) and $W \subset \mathbb{R}^n \times \mathbb{R}^m$ open neighborhood of $F(a,b) = (a,0)$, such that $F : A \times B \to W$ has a differentiable inverse $h : W \to A \times B$. Since F is the identity in the first n coordinates we also have $h(x,y) = (x, k(x,y))$ for some differentiable function $k : W \to B$.

Let $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m, (x,y) \mapsto y$ and thus $\pi \circ F = f$. This yields

$$f(x,k(x,y)) = (f \circ h)(x,y) = ((\pi \circ F) \circ h)(x,y) = (\pi \circ (F \circ h))(x,y) = \pi(x,y) = y\,.$$

Thus $f(x,k(x,0)) = 0$ and we can define $g(x) : A \to B$ by $g(x) := k(x,0)$. □

Remark 5.0.2: (1) Note that instead of saying that $\det(M) \neq 0$ we could equivalently ask that the linear map $L : \mathbb{R}^m \to \mathbb{R}^m, y \mapsto Df(a,b)(0,y)$ is invertible.

(2) It is not completely natural to ask that the $m \times m$ matrix M of the last m columns of the Jacobi matrix $\partial f(a,b)$ is invertible. The more natural assumption is that $Df(a,b)$ (equivalently $\partial f(a,b)$) has *full rank*, i.e. rank m. Thus after relabelling the coordinates of $\mathbb{R}^n \times \mathbb{R}^m$ (i.e. switching column vectors in $\partial f(a,b)$) we can assume that the $m \times m$ matrix M of the last m columns of the $\partial f(a,b)$ is invertible, and we are in the setup above.

(3) We have from the above theorem that $f(x,g(x)) = 0$ for all $x \in A$. Differentiating this equation yields for $i \in \{1,\ldots,m\}, j \in \{1,\ldots,n\}$ by the chain rule

$$0 = \partial_j(f_i(x,g(x))) = \partial_j f_i(x,g(x)) + \sum_{l=1}^{m} \partial_{n+l} f_i(x,g(x)) \partial_j g_l(x)\,.$$

We can write this as

$$M \cdot \begin{pmatrix} \partial_j g_1(x) \\ \vdots \\ \partial_j g_m(x) \end{pmatrix} = - \begin{pmatrix} \partial_j f_1(x,g(x)) \\ \vdots \\ \partial_j f_m(x,g(x)) \end{pmatrix}$$

and thus since M is invertible

$$\begin{pmatrix} \partial_j g_1(x) \\ \vdots \\ \partial_j g_m(x) \end{pmatrix} = -M^{-1} \cdot \begin{pmatrix} \partial_j f_1(x,g(x)) \\ \vdots \\ \partial_j f_m(x,g(x)) \end{pmatrix}$$

Note that since f is continuously differentiable, this implies that g is continuously differentiable as well (since g is continuous). One can furthermore show that if $f \in C^k(\mathbb{R}^{n+m};\mathbb{R}^m)$ then $g \in C^k(\mathbb{R}^n;\mathbb{R}^m)$.

5.1 Level sets and submanifolds of Euclidean Space

Let $U \subset \mathbb{R}^{n+m}$ be open and $f : U \to \mathbb{R}^m$ be continuously differentiable. For $c \in \mathbb{R}^m$ consider the level set

$$f^{-1}(c) := \{x \in \mathbb{R}^{n+m} \,|\, f(x) = c\}\,.$$

Definition 5.1.1 (Regular Value)**.** *We say that c is a regular value of f provided $Df(x)$ has full rank (i.e. rank m) for all $x \in f^{-1}(c)$.*

Remark 5.1.2: (1) Note that if $f^{-1}(c) = \emptyset$, then we also call c a regular value of f.

(2) For $U \subset \mathbb{R}^{n+1}$ open and $f : U \to \mathbb{R}$, i.e. $m = 1$ in the above setup, then the condition that c is a regular value is equivalent to $\nabla f(x) \neq 0$ for all $x \in f^{-1}(c)$.

We see that the Implicit Function Theorem and Remark 5.0.2 (2) directly implies:

Proposition 5.1.3. *Let $U \subset \mathbb{R}^n \times \mathbb{R}^m$ be open and $f : U \to \mathbb{R}^m$ be continuously differentiable. Let $c \in \mathbb{R}^m$ be a regular value of f. Then for every $x \in S := f^{-1}(c)$ there is an open neighborhood V of x such that $S \cap V$ can be written as the graph of a continuously differentiable function $g : A \to \mathbb{R}^m$ (where $A \subset \mathbb{R}^n$ open, after suitably relabelling the coordinates).*

On the other hand consider $A \subset \mathbb{R}^n$ open and $g : A \to \mathbb{R}^m$ continuously differentiable. Let $S := \text{graph}(g) \subset A \times \mathbb{R}^m$. Writing $z \in \mathbb{R}^n \times \mathbb{R}^m$ again as $z = (x, y)$ we can consider the function $G : A \times \mathbb{R}^m, (x, y) \mapsto y - g(x)$. Note that $DG(z)$ has rank m for all $z \in A \times \mathbb{R}^m$ and $\text{graph}(g) = G^{-1}(0)$. Thus the graph of a function can always be written as the level set of a regular value of a function. This motivates the following definition.

Definition 5.1.4 (Submanifolds of Euclidean space)**.** *Let $S \subset \mathbb{R}^n$ and $0 \leq k \leq n-1$. S is called a k-dimensional submanifold of $\mathbb{R}^n$ if for each $x \in S$ there exists $U \subset \mathbb{R}^n$ an open neighborhood of x and $f : U \to \mathbb{R}^{n-k}$ such 0 is a regular value of f and $S \cap U = f^{-1}(0)$.*

Note that by Proposition 5.1.3 this is equivalent to asking that for each $x \in S$ there exists $U \subset \mathbb{R}^n$ an open neighborhood of x such that $S \cap U$ can be written as the graph of a function from an open set in $\mathbb{R}^k$ to $\mathbb{R}^{n-k}$.

Remark 5.1.5 (Tangent plane to a graph)**:** Consider again $A \subset \mathbb{R}^n$ open and $g : A \to \mathbb{R}^m$ continuously differentiable. Let $S := \text{graph}(g) \subset A \times \mathbb{R}^m$. Recall that as discussed in Section 3.4.2 and Section 3.4.3 (extended here to all dimensions) the tangent plane to S at $a = (x, g(x))$ is the affine plane $a + T_aS$ where T_aS is the vector space spanned by the n linearly independent vectors

$$e_1 + \sum_{j=1}^m \partial_1 g_j e_{n+j}\ ,\ e_2 + \sum_{j=1}^m \partial_2 g_j e_{n+j},\ \cdots\ ,\ e_n + \sum_{j=1}^m \partial_n g_j e_{n+j}\,.$$

Assume we can also write $S \cap U = f^{-1}(0)$ where 0 is the regular value of a continuously differentiable function $f : U \to \mathbb{R}^m$ and $U \subset \mathbb{R}^n \times \mathbb{R}^m$ an open neighborhood of a. Since $f(x, g(x)) = 0$ we can differentiate to see that

$$Df(a)\Big(e_i + \sum_{j=1}^{m} \partial_i g_j e_{n+j}\Big) = 0$$

for all $i = 1, \dots, n$. But this implies that

$$T_a S \subset \ker Df(a)\,.$$

Furthermore, since a is a regular value of f and thus $Df(a) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ has full rank, we have by rank-nullity that $\dim \ker Df(a) = n$. Since $\dim T_a S = n$ we have

$$T_a S = \ker Df(a)\,.$$

6 Second order derivatives

6.1 The Hessian

For $U \subset \mathbb{R}^n$ open, recall that if $f\colon U \to \mathbb{R}$ is differentiable at x then $Df(x) \in L(\mathbb{R}^n, \mathbb{R}) =: (\mathbb{R}^n)^*$.

$(\mathbb{R}^n)^*$ is the space of linear funtionals on $\mathbb{R}^n$ also called the dual space of $\mathbb{R}^n$. Note that if $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$ then the standard basis of $(\mathbb{R}^n)^*$ is given by the linear maps $\omega_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \dots, n$ defined via

$$\omega_i(e_j) = \delta_{ij}\,.$$

(Note that we can also write $\omega_i(\cdot) = \langle e_i, \cdot \rangle$, using that the standard basis is orthonormal). We can thus write any $L \in (\mathbb{R}^n)^*$ as

$$L = \sum_{i=1}^n L(e_i)\omega_i\,.$$

Using the basis $\{\omega_1, \dots, \omega_n\}$ we can thus identify $(\mathbb{R}^n)^*$ again with $\mathbb{R}^n$. Note that if in coordinates we identify elements in $\mathbb{R}^n$ with column vectors, i.e. elements in $\mathbb{R}^{n,1}$, then elements in $(\mathbb{R}^n)^*$ can be identified with row vectors, i.e. elements in $\mathbb{R}^{1,n}$. So in coordinates if $h \in \mathbb{R}^n$ and $L \in (\mathbb{R}^n)^*$, i.e.

$$h = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} l_1 \cdots l_n \end{pmatrix}$$

Then

$$L(h) = \sum_{i=1}^n L(e_i)\omega_i(h) = \sum_{i=1}^n l_i\omega_i(h) = \sum_{i=1}^n l_i h_i = \begin{pmatrix} l_1 \cdots l_n \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}.$$

Now assume that $f \in C^1(U)$, i.e. the map $Df\colon U \to (\mathbb{R}^n)^*$ given by $x \mapsto Df(x)$ is continuous. Suppose further that Df is differentiable at $x \in U$, i.e. there exists $H \in L(\mathbb{R}^n, (\mathbb{R}^n)^*)$ such that

$$Df(x+h) = Df(x) + H(h) + R(x,h) \tag{6.1}$$

where

$$\lim_{\substack{h\to 0\\ h\neq 0}} \frac{\|R(x,h)\|_{\mathrm{op}}}{\|h\|} = 0. \tag{6.2}$$

We denote H, if it exists, by $D^2 f(x)$.

Note that for $h \in \mathbb{R}^n$ then $H(h) \in (\mathbb{R}^n)^*$ (and is linear in h) so we can apply it to another vector $k \in \mathbb{R}^n$ (and it is linear in k). Thus we can see $D^2 f(x)$ as a bilinear form on $\mathbb{R}^n$, i.e.

$$D^2 f(x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, (h,k) \mapsto D^2 f(x)(h,k)\,.$$

Note that in coordinates

$$Df(x) = \sum_{i=1}^n Df(x)(e_i)\omega_i = \sum_{i=1}^n \partial_i f(x)\omega_i$$

or in terms of matrices $\partial f \colon U \to \mathbb{R}^{1,n}$, that is,

$$x \mapsto \partial f(x) = \begin{pmatrix} \partial_1 f(x) & \cdots & \partial_n f(x) \end{pmatrix}.$$

That is we can see $x \mapsto \partial_i f(x)$ as the coordinate functions of the map $x \mapsto Df(x)$. Thus if $D^2 f(x) \in L(\mathbb{R}^n, (\mathbb{R}^n)^*)$ exists then by Theorem 3.3.1 we have that the second partials of f exist at x and

$$D^2 f(x)(e_i, e_j) = \partial^2_{ji} f(x) := \big(\partial_j(\partial_i f)\big)(x)\,.$$

In coordinates, for $h, k \in \mathbb{R}^n$, i.e.

$$h = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \quad \text{and} \quad k = \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix},$$

we have

$$D^2 f(x)(h,k) = D^2 f(x)\Bigg(\sum_{i=1}^n h_i e_i\,, \sum_{j=1}^n k_j e_j\Bigg) = \sum_{i,j=1}^n D^2 f(x)(e_i, e_j) h_i k_j = \sum_{i,j=1}^n \partial^2_{ji} f(x) h_i k_j\,,$$

which we can write as

$$D^2 f(x)(h,k) = h^T\big(\partial^2 f(x)\big)k = \begin{pmatrix} h_1 & \cdots & h_n \end{pmatrix} \begin{pmatrix} \partial^2_{11} f(x) & \cdots & \partial^2_{1n} f(x) \\ \vdots & & \vdots \\ \partial^2_{n1} f(x) & \cdots & \partial^2_{nn} f(x) \end{pmatrix} \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix},$$

where we define the $n \times n$ matrix

$$\partial^2 f(x) := \begin{pmatrix} \partial^2_{11} f(x) & \cdots & \partial^2_{1n} f(x) \\ \vdots & & \vdots \\ \partial^2_{n1} f(x) & \cdots & \partial^2_{nn} f(x) \end{pmatrix}. \tag{6.3}$$

The matrix $\partial^2 f(x)$ is called the *Hessian* of f at x and it is also denoted by $\operatorname{Hess} f(x)$. It is also common to write

$$\operatorname{Hess} f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \dots & \frac{\partial^2 f}{\partial x_n^2}(x) \end{pmatrix}.$$

As we shall see below, even if all the second order partials $\frac{\partial^2 f}{\partial x_j \partial x_i}(x)$ exist, (6.1) (together with (6.2)) may still not hold.

6.2 Commutativity of second order partial derivatives

Proposition 6.2.1 ($D^2 f$ exists implies that second order partial derivatives at x commute)**.** *Let $U \subset \mathbb{R}^2$ be open and $f \in C^1(U)$. For $(x,y) \in U$ suppose that $D^2 f(x,y)$ exists. Then $\partial_{12}^2 f(x,y) = \partial_{21}^2 f(x,y)$.*

Remark 6.2.2: Note that if $U \subset \mathbb{R}^n$ open and $f \in C^1(U)$, $a \in U$ and $D^2 f(a)$ exists, then for $i, j \in \{1, \dots n\}, i < j$ we can consider for $(h,k) \in \mathbb{R}^2$ with $\|(h,k)\| < \varepsilon$ and $\varepsilon > 0$ sufficiently small

$$(h,k) \mapsto \tilde{f}(h,k) := f(a_1, \dots, a_i + h, \dots, a_j + k, \dots, a_n)$$

and apply the above statement to $\tilde{f}$ at $(x,y) = (0,0)$ to obtain

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(a) = \frac{\partial^2 f}{\partial x_j \partial x_i}(a) \qquad \forall\, i,j \in \{1, \dots, n\},$$

i.e. $\operatorname{Hess} f(a)$ is a symmetric matrix. One can also phrase this in saying that $D^2 f(a)(h,k)$ is a *symmetric* bilinear form.

Proof of Proposition 6.2.1. Pick $\delta > 0$ so that $(x+h, y+k) \in U$ if $|h| < \delta$ and $|k| < \delta$. Define

$$\sigma(h,k) := f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x,y)$$

Fix $k \in (-\delta, \delta)$ and set

$$\eta_k(s) := f(x+s, y+k) - f(x+s, y)\,.$$

Observe that $\sigma(h,k) = \eta_k(h) - \eta_k(0)$. So, by the Mean Value Theorem for real valued functions of a single variable, $\exists\ \theta_1 \in (0,1)$ such that

$$\sigma(h,k) = \eta_k'(\theta_1 h)\, h = \Big(\partial_1 f(x + \theta_1 h, y+k) - \partial_1 f(x + \theta_1 h, y)\Big) h. \tag{6.4}$$

Similarly, fix $h \in (-\delta, \delta)$ and set

$$\xi_h(t) := f(x+h, y+t) - f(x, y+t)\,.$$

Then $\sigma(h,k) = \xi_h(k) - \xi_h(0)$ and $\exists\ \theta_2 \in (0,1)$ such that

$$\sigma(h,k) = \xi_h'(\theta_2 k)\, k = \Big(\partial_2 f(x+h, y + \theta_2 k) - \partial_2 f(x, y + \theta_2 k)\Big) k. \tag{6.5}$$

We now use the assumption that $D^2 f(x,y)$ exists and write (6.1) in the following matrix form:

$$\begin{aligned}\big(\partial_1 f(x+h,y+k) \quad \partial_2 f(x+h,y+k)\big) &= \big(\partial_1 f(x,y) \quad \partial_2 f(x,y)\big) \\ &\quad + \begin{pmatrix} h & k \end{pmatrix}\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} + \big(r_1(h,k) \quad r_2(h,k)\big)\end{aligned} \tag{6.6}$$

where

$$\partial^2 f(x,y) := \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}$$

and

$$\lim_{\substack{(h,k)\to(0,0)\\(h,k)\neq(0,0)}} \frac{|r_i(h,k)|}{\|(h,k)\|} = 0 \quad \text{for } i \in \{1,2\} \tag{6.7}$$

Using (6.6), (6.4) and (6.5) can be rewritten as

$$\begin{aligned}\sigma(h,k) &= \big((\alpha\theta_1 h + bk) - \alpha\theta_1 h + r_1(\theta_1 h, k) - r_1(\theta_1 h, 0)\big)h, \\ \sigma(h,k) &= \big((ah + \beta\theta_2 k) - \beta\theta_2 k + r_2(h, \theta_2 k) - r_2(0, \theta_2 k)\big)k.\end{aligned}$$

Setting $h = k$ and equating the right hand sides of the above equations gives

$$(b-a)h^2 = \big(r_2(h,\theta_2 h) - r_2(0,\theta_2 h) - r_1(\theta_1 h, h) + r_1(\theta_1 h, 0)\big)h.$$

Dividing both sides by $h^2 \neq 0$, taking the limit as $h \to 0$ and using (6.7) yields $a = b$ as required. □

Corollary 6.2.3. *Assume that $U \subset \mathbb{R}^n$ is open, $f \in C^1(U)$ and assume* $\operatorname{Hess} f(x)$ *exists and is continuous on U. Then for all $x \in U$ the matrix* $\operatorname{Hess} f(x)$ *is symmetric (i.e. all second order partial derivatives of f at x commute).*

Proof. The continuity of $x \mapsto \operatorname{Hess} f(x)$ together with Theorem 3.3.4 yields that $x \mapsto Df(x)$ is differentiable for all $x \in U$. Proposition 6.2.1 then yields the statement. □

Definition of C^k spaces. Recall the definition of C^0 and C^1

$$\begin{aligned}C^0(U,\mathbb{R}^k) &:= \{f\colon U \to \mathbb{R}^k \mid f \text{ is continuous}\} \\ C^1(U,\mathbb{R}^k) &:= \{f\colon U \to \mathbb{R}^k \mid \partial f\colon U \to \mathbb{R}^{k,n} \text{ is continuous}\}.\end{aligned}$$

Similarly, we define

$$\begin{aligned}C^2(U) = C^2(U,\mathbb{R}) &:= \{f \in C^1(U) \mid \partial^2 f\colon U \to \mathbb{R}^{n,n} \text{ is continuous}\} \\ \text{and } C^2(U,\mathbb{R}^k) &:= \{f\colon U \to \mathbb{R}^k \mid f_1,\dots,f_k \in C^2(U)\}, \text{ where } f = (f_1,\dots,f_k).\end{aligned}$$

If we wish to consider the third derivative of $f\colon U \to \mathbb{R}$, we identify $\mathbb{R}^{n,n}$ with $\mathbb{R}^{n^2}$ and proceed as for the second derivative of f. However, we shall have no need for this. Nevertheless, we define:

$$C^k(U,\mathbb{R}^m) = \{f\colon U \to \mathbb{R}^m : \text{ all partial derivatives of } f \text{ up to, and including, order } k \text{ exist and are continuous on } U\}.$$

$C^k(U,\mathbb{R})$ is abbreviated to just $C^k(U)$. We say that $u \in C^\infty(U)$ if $u \in C^k(U)$ for all $k \in \mathbb{N}\cup\{0\}$.

Example 6.2.4 (Second order partial derivatives may not commute)**:** Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = xy\left(\frac{x^2-y^2}{x^2+y^2}\right) \quad \text{if } (x,y) \neq (0,0), \qquad f(0,0)=0.$$

We shall show below that

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = 1 \qquad \text{but} \qquad \frac{\partial^2 f}{\partial y \partial x}(0,0) = -1. \tag{6.8}$$

Remark 6.2.5: Using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, f can be written as $xy\cos(2\theta)$ which is approximately equal to xy near the x-axis ($\theta = 0$) but is approximately equal to $-xy$ near the y-axis ($\theta = \pi/2$). This explains (6.8).

Formal Proof of (6.8). On observing that

$$\frac{x^2-y^2}{x^2+y^2} = 1 - \frac{2y^2}{x^2+y^2} = \frac{2x^2}{x^2+y^2} - 1$$

we easily see that

$$\frac{\partial f}{\partial x} = y\left(\frac{x^2-y^2}{x^2+y^2}\right) + xy\left(\frac{4y^2x}{(x^2+y^2)^2}\right) = y\left(\frac{x^2-y^2}{x^2+y^2} + \frac{4x^2y^2}{(x^2+y^2)^2}\right) \qquad \text{if } (x,y)\neq(0,0)$$

and that, similarly,

$$\frac{\partial f}{\partial y} = x\left(\frac{x^2-y^2}{x^2+y^2} - \frac{4x^2y^2}{(x^2+y^2)^2}\right) \qquad \text{if } (x,y)\neq(0,0).$$

Furthermore,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x} = 0$$

and

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y} = 0.$$

Therefore,

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{x\to 0}\frac{\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)}{x} = \lim_{x\to 0}\frac{x}{x} = 1$$

and

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{y\to 0} \frac{\frac{\partial f}{\partial x}(0,y) - \frac{\partial f}{\partial x}(0,0)}{y} = \lim_{y\to 0} \frac{-y}{y} = -1$$

thereby verifying (6.8).

Remark 6.2.6: For f in the preceding example, $D^2 f(0,0)$ does not exist, even though all the second order partials do. For if it did, then by the preceding proposition, the mixed second order partials would have to commute, which they do not.

6.3 Second order Taylor expansion

Recall that for $f \in C^2((b,c))$ we have the second order Taylor expansion around $a \in (b,c) \subset \mathbb{R}$ and $h \in \mathbb{R}$ such that $a + h \in (b,c)$

$$f(a+h) = f(a) + f'(a)h + \tfrac{1}{2} f''(a)h^2 + R(a,h)$$

where

$$\lim_{\substack{h\to 0\\ h\neq 0}} \frac{|R(a,h)|}{h^2} = 0\,.$$

Remark 6.3.1: In the usual Taylor expansion, if one makes the stronger assumption that $f \in C^3((b,c))$, one can show that for some $\theta \in (0,1)$ (depending on h)

$$R(a,h) = f'''(a+\theta h)h^3\,.$$

We will show a higher dimensional analogue of the above result.

Theorem 6.3.2 (Second order Taylor expansion)**.** *Let $U \subset \mathbb{R}^n$ be open, convex and $a \in U$. Then for $f \in C^2(U)$ and $h \in \mathbb{R}^n$ such that $a + h \in U$ it holds*

$$\begin{aligned} f(a+h) &= f(a) + Df(a)(h) + \tfrac{1}{2} D^2 f(a)(h,h) + R(a,h) \\ &= f(a) + \sum_{i=1}^{n} \partial_i f(a) h_i + \frac{1}{2} \sum_{i,j=1}^{n} \partial_{ij}^2 f(a) h_i h_j + R(a,h) \end{aligned}$$

where

$$\lim_{\substack{h\to 0\\ h\neq 0}} \frac{|R(a,h)|}{\|h\|^2} = 0\,.$$

Proof. Since U is convex we have $a + th \in U$ $\forall\, t \in [0,1]$ and, keeping h fixed, we can define

$g\colon [0,1] \to \mathbb{R}$ by

$$g(t) := f(a+th) + (1-t)\sum_{i=1}^{n} \partial_i f(a+th)h_i + \frac{1}{2}(1-t)^2 \sum_{i,j=1}^{n} \partial_{ij}^2 f(a)h_i h_j.$$

Note that by the chain rule $g \in C^0([0,1]) \cap C^1((0,1))$ and

$$g(1) = f(a+h) \qquad \text{and} \qquad g(0) = f(a) + \sum_{i=1}^{n} \partial_i f(a)h_i + \frac{1}{2}\sum_{i,j=1}^{n} \partial_{ij}^2 f(a)h_i h_j\,. \tag{6.9}$$

By the Mean Value Theorem for real valued functions of a single real variable there exists $\theta \in (0,1)$ such that

$$g(1) - g(0) = g'(\theta). \tag{6.10}$$

By the chain rule,

$$\begin{aligned} g'(t) &= \sum_{i=1}^{n} \partial_i f(a+th)h_i - \sum_{i=1}^{n} \partial_i f(a+th)h_i + (1-t)\sum_{i,j=1}^{n} \partial_{ij}^2 f(a+th)h_i h_j \\ &\qquad - (1-t)\sum_{i,j=1}^{n} \partial_{ij}^2 f(a)h_i h_j \qquad (6.11) \\ &= (1-t)\sum_{i,j=1}^{n} \big(\partial_{ij}^2 f(a+th) - \partial_{ij}^2 f(a)\big)h_i h_j. \end{aligned}$$

By continuity of $\partial^2 f$ at a, given $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$\|h\| < \delta \ \Rightarrow\ |\partial_{ij}^2 f(a+th) - \partial_{ij}^2 f(a)| < \varepsilon \quad \forall\, i,j \in \{1,\dots,n\} \quad \text{and} \quad \forall\, t \in [0,1]. \tag{6.12}$$

Set $R(a,h) := g'(\theta)$. Then, substituting (6.9) and (6.11) in (6.10) we see that

$$f(a+h) = f(a) + \sum_{i=1}^{n} \partial_i f(a)h_i + \frac{1}{2}\sum_{i,j=1}^{n} \partial_{ij}^2 f(a)h_i h_j + R(h)$$

For $\|h\| < \delta$ we have from (6.12) that $|R(a,h)| \leqslant \varepsilon n^2 \|h\|^2$ and thus

$$\lim_{\substack{h\to 0\\ h\neq 0}} \frac{|R(a,h)|}{\|h\|^2} = 0\,.$$

□

6.4 Critical points, local maxima and minima, saddles

6.4.1 Critical points

Definition 6.4.1. *Let $U \subset \mathbb{R}^n$ be open and $f \in C^1(U)$. A point $p \in U$ is calles a* critical point *of f if $\nabla f(p) = 0$.*

Recall that Lemma 4.0.2 implies that if f has a local maximum or minimum at p then $\nabla f(p) = 0$.

We will now try to understand the Hessian at a critical point p of f to deduce if p is a local maximum or minimum or neither. We first need some further linear algebra.

Recall that a symmetric matrix $P \in \mathbb{R}^{n,n}$ is

(i) *positive definite* if $x^T P x = \langle x, Px\rangle > 0 \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}$.

(ii) *positive semidefinite* if $x^T P x = \langle x, Px\rangle \geq 0 \quad \forall\, x \in \mathbb{R}^n$.

(iii) *negative definite* if $x^T P x = \langle x, Px\rangle < 0 \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}$.

(iv) *negative semidefinite* if $x^T P x = \langle x, Px\rangle \leq 0 \quad \forall\, x \in \mathbb{R}^n$.

(v) *indefinite* if there exist $x, y \in \mathbb{R}$ such that $x^T P x > 0$ and $y^T P y < 0$.

Diagonalisation of symmetric matrices. Recall from Advanced Linear Algebra that every symmetric matrix can be diagonalised by an orthogonal matrix, i.e. there exist real eigenvalues $\lambda_1, \dots, \lambda_n$ corresponding to an orthonormal set of eigenvectors $\{v_1, \dots, v_n\}$. Thus with respect to this basis of eigenvectors $P = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. We can write this in the form that there exists an orthogonal matrix O (the matrix corresponding to the change of basis $\{v_1, \dots, v_n\} \to \{e_1, \dots, e_n\}$) such that

$$O^T P O = \operatorname{diag}(\lambda_1, \dots, \lambda_n)\,.$$

Proposition 6.4.2. *If we arrange the eigenvalues of P in an increasing order, i.e. $\lambda_1 \leq \dots \leq \lambda_n$, then for all $x \in \mathbb{R}^n$*

$$\lambda_1 \|x\|^2 \leq \langle x, Px\rangle \leq \lambda_n \|x\|^2\,.$$

Proof. Let $\{v_1, \dots, v_n\}$ be the orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of P, i.e. $Pv_i = \lambda_i v_i$. For $x \in \mathbb{R}^n$ let $a_i := \langle v_i, x\rangle$. Then $x = \sum_{i=1}^n a_i v_i$ and $\|x\|^2 = \sum_{i=1}^n a_i^2$. Thus

$$\langle x, Px\rangle = \sum_{i=1}^n (a_i)^2 \lambda_i \geq \lambda_1 \sum_{i=1}^n (a_i)^2 = \lambda_1 \|x\|^2$$

and similarly

$$\langle x, Ps\rangle = \sum_{i=1}^{n}(a_i)^2\lambda_i \leq \lambda_n \sum_{i=1}^{n}(a_i)^2 = \lambda_n\|x\|^2 .$$

□

6.4.2 Second order derivative test

We will now discuss how we can use information on the Hessian to deduce if a critical point is a local maximum, local minimum or a saddle point.

Proposition 6.4.3. *Let $U \subset \mathbb{R}^n$ be open and $f \in C^2(U)$. Assume that $\nabla f(p) = 0$ for some $p \in U$.*

(i) If Hess $f(p)$ *is positive definite then f has a strict local minimum at p.*

(ii) If Hess $f(p)$ *is negative definite then f has a strict local maximum at p.*

(iii) If Hess $f(p)$ *is indefinite then f has neither a local minimum nor a local maximum at p and p is called a* saddle point*. (If* Hess $f(p)$ *is indefinite and has a zero eigenvalue then p is called a* degenerate *saddle point. However, we shall not distinguish between nondegenerate and degenerate saddle points, and we shall refer to both of them as just saddle points.)*

(iv) If Hess $f(p)$ *is positive or negative semidefinite but not definite then the test is inconclusive, i.e. f may have a minimum at p, or a maximum or a saddle point.*

Example 6.4.4 (Simplest examples to illustrate the second derivative test)**:**

(i) $f(x, y) = x^2 + y^2$. f has a minimum at $(0, 0)$, $\nabla f(0, 0) = 0$ and Hess $f(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is positive definite.

(ii) $f(x, y) = -x^2 - y^2$. f has a maximum at $(0, 0)$, $\nabla f(0, 0) = 0$ and Hess $f(0, 0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ is negative definite.

(iii) $f(x, y) = x^2 - y^2$. f has a saddle point at $(0, 0)$, $\nabla f(0, 0) = 0$ and Hess $f(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ is indefinite.

(iv) $f(x, y) = x^2 + y^4$ has a strict minimum at $(0, 0)$, $g(x, y) = -x^2 - y^4$ has a strict maximum at $(0, 0)$, $f(x, y) = x^2 - y^4$ has a saddle point at $(0, 0)$, $f(x, y) = x^2$ has a nonstrict minimum at $(0, 0)$, $g(x, y) = -x^2$ has a nonstrict maximum at $(0, 0)$. All these functions have a gradient that vanishes at $(0, 0)$ and a semidefinite Hessian at $(0, 0)$. Explicitly,

$$\text{Hess}\, f(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \text{Hess}\, g(0, 0) = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof of Proposition 6.4.3. Set $P := \text{Hess}\, f(p)$. Then, by Taylor's second order expansion (Theorem 6.3.2),

$$f(p + h) = f(p) + \tfrac{1}{2}\langle h, Ph\rangle + R(p, h).$$

(i) If P is positive definite, then its smallest eigenvalue $\lambda_1 > 0$ and there exists $\delta > 0$ such that

$$\begin{aligned} 0 < \|h\| < \delta \ &\Rightarrow \ |R(h)| \leqslant \tfrac{1}{4}\lambda_1 \|h\|^2 \\ &\Rightarrow \ f(p+h) \geqslant f(p) + \tfrac{1}{2}\lambda_1\|h\|^2 - \tfrac{1}{4}\lambda_1\|h\|^2 > f(p), \end{aligned}$$

i.e. f has a strict local minimum at p.

(ii) If P is negative definite, then its largest eigenvalue $\lambda_n < 0$. There exists $\delta > 0$ such that

$$\begin{aligned} 0 < \|h\| < \delta \ &\Rightarrow \ |R(h)| \leqslant -\tfrac{1}{4}\lambda_n \|h\|^2 \\ &\Rightarrow \ f(p+h) \leqslant f(p) + \tfrac{1}{2}\lambda_n\|h\|^2 - \tfrac{1}{4}\lambda_n\|h\|^2 < f(p), \end{aligned}$$

i.e. f has a strict local maximum at p.

(iii) If P is indefinite, then its smallest eigenvalue λ_1 must be negative and its largest eigenvalue λ_n must be positive. Set $g_1(t) := f(p + tv_1)$, t sufficiently small. Then $g_1(0) = f(p)$, $g_1'(0) = \langle \nabla f(p), v_1\rangle = 0$ and $g_1''(0) = \langle v_1, Pv_1\rangle < 0$ and therefore, g_1 has a strict maximum at p.

Similarly, set $g_n(t) := f(p + tv_n)$, t sufficiently small. Then $g_n(0) = f(p)$, $g_n'(0) = 0$ and $g_n''(0) = \langle v_n, Pv_n\rangle > 0$ and therefore, g_n has a strict minimum at p. We have shown that f has a saddle point at p.

(iv) Let

$$f_+(x,y) := x^4 + y^4, \qquad f_-(x,y) := -x^4 - y^4, \qquad f(x,y) := x^4 - y^4.$$

f_+ has a strict local minimum at $(0,0)$, f_- has a strict local maximum at $(0,0)$ and f has a saddle point at $(0,0)$. Yet, all these functions have a gradient and Hessian that vanish at $(0,0)$. Therefore no conclusion can be drawn about a critical point p of a function whose Hessian at p is positive or negative semidefinite. See also the functions in (iv) of the preceding set of examples.

□

Definitiveness test for 2×2 symmetric matrices. For a symmetric matrix $P \in \mathbb{R}^{2,2}$ it is possible to gain information on the definiteness of P by looking at its determinant. More precisely

A 2×2 symmetric matrix $P = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ is

(i) positive definite if $\det P = ac - b^2 > 0$ *and* $a > 0$ *or* $c > 0$.

(ii) negative definite if $\det P > 0$ *and* $a < 0$ *or* $c < 0$.

(iii) indefinite if $\det P < 0$.

(iv) semidefinite if $\det P = 0$.

Proof. Let λ_1 and λ_2 be the two real eigenvalues of P. Then, considering the characteristic polynomial of P

$$\begin{aligned}(a-\lambda)(c-\lambda)-b^2 &= \lambda^2-(a+c)\lambda+ac-b^2\\ &= (\lambda-\lambda_1)(\lambda-\lambda_2)=\lambda^2-(\lambda_1+\lambda_2)\lambda+\lambda_1\lambda_2,\end{aligned}$$

we see that

$$ac-b^2=\lambda_1\lambda_2, \qquad a+c=\lambda_1+\lambda_2.$$

(i) If $ac-b^2>0$, then $ac>0$ and therefore $a>0$ iff $c>0$. It follows that

$$ac-b^2>0 \quad \text{and } a>0 \text{ or } c>0 \quad \Leftrightarrow \quad \lambda_1,\lambda_2>0 \quad \Leftrightarrow \quad P \text{ is positive definite.}$$

(ii) If $ac-b^2>0$, then $a<0$ iff $c<0$ and therefore,

$$ac-b^2>0 \quad \text{and } a<0 \text{ or } c<0 \quad \Leftrightarrow \quad \lambda_1,\lambda_2<0 \quad \Leftrightarrow \quad P \text{ is negative definite.}$$

(iii) $ac-b^2<0 \Leftrightarrow \lambda_1,\lambda_2$ have opposite signs $\Leftrightarrow P$ is indefinite.

(iv) $ac-b^2=0 \Leftrightarrow$ at least one of λ_1 and λ_2 must vanish $\Leftrightarrow P$ is positive or negative semidefinite.

□

Example 6.4.5: Consider $f(x,y)=x^3-3x\sin y, \quad x\in\mathbb{R},\ y\in(-\pi/4,3\pi/4)$. Classify the three criticial points of f.

Solution. We have

$$\nabla f(x,y)=(3x^2-3\sin y,-3x\cos y).$$

At a critical point, *both* equations $x^2-\sin y=0$ and $x\cos y=0$ must hold. Starting with the second equation we see that either $x=0$ or $y=\pi/2$.

Looking at the first equation, if $x=0$ then $\sin y=0$ and therefore, $y=0$ since y has to lie in $(-\pi/4,3\pi/4)$. So f has a critical point at $(0,0)$.

Again looking at the first equation, if $y=\pi/2$ then $x^2=1$ and therefore, $x=1$ or $x=-1$. So f has two more critical points at $(1,\pi/2)$ and at $(-1,\pi/2)$.

$$\operatorname{Hess} f(x,y)=\begin{pmatrix}6x & -3\cos y\\ -3\cos y & 3x\sin y\end{pmatrix}.$$

$$\text{Hess}\, f(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix} \quad \text{which is indefinite, and therefore } f \text{ has a saddle point at } (0,0).$$

$$\text{Hess}\, f(1,\pi/2) = \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix} \quad \text{which is positive definite, and therefore}$$

$$f \text{ has a strict local minimum at } (1,\pi/2).$$

$$\text{Hess}\, f(-1,\pi/2) = \begin{pmatrix} -6 & 0 \\ 0 & -3 \end{pmatrix} \quad \text{which is negative definite, and therefore}$$

$$f \text{ has a strict local maximum at } (1,\pi/2).$$

7 Integration

In this section we will discuss Riemann intergration in higher dimensions.

7.1 Basic definitions

Definition 7.1.1. *(i) Let $a, b \in \mathbb{R}^n$ be s.t. $a_i < b_i$ for all $i \in \{1, \ldots, n\}$. We call the set*

$$R^{a,b} := \{x \in \mathbb{R}^n \,|\, a_i \leq x_i \leq b_i \textit{ for all } i \in \{1, \ldots, n\}\}$$

a rectangle. Note that this a natural extension of the notion of an interval.

(ii) Recall that a partition *of an interval $[c, d] \subset \mathbb{R}$ is a (finite) sequence $t_0, \ldots, t_k$ s.t. $c = t_0 \leq t_1 \leq \cdots \leq t_k = d$. This divides $[c, d]$ into k subintervals $[t_{i-1}, t_i]$ for $i = 1, \ldots, k$.*

A partition of a rectangle $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is a collection $P = (P_1, \ldots, P_n)$ where each P_i is a partition of $[a_i, b_i]$. Assume P_i is a partition of $[a_i, b_i]$ into N_i subintervals of $[a_i, b_i]$. Then P subdivides P into $N = N_1 \cdots N_n$ subrectangles. We call this the subrectangles *of the partition P.*

Let $A \subset \mathbb{R}^n$ be a rectangle, $f : A \to \mathbb{R}$ and P a partition of A. Then for each subrectangle S of P let

$$m_S(f) := \inf\{f(x) \,|\, x \in S\} \qquad M_S(f) := \sup\{f(x) \,|\, x \in S\}$$

and let

$$v(S) := (q_1 - p_1) \cdots (q_n - p_n)$$

be the volume *of $S = R^{p,q}$. We define the* lower sum *and* upper sum *of f w.r.t. P as*

$$L(f, P) := \sum_{S \in P} m_S(f) v(S) \qquad U(f, P) := \sum_{S \in P} M_S(f) v(S)\,.$$

It clearly holds that $L(f, P) \leq U(f, P)$.

Lemma 7.1.2. *Suppose the partition P' refines P (that is each subrectangle of P' is contained in a subrectangle of P), then*

$$L(f, P) \leq L(f, P') \quad \textit{and} \quad U(f, P') \leq U(f, P)\,.$$

Proof. Each subrectangle S of P is divided into $S_1, \ldots, S_\alpha$ subrectangles of P', so $v(S) = v(S_1) + \cdots + v(S_\alpha)$. Now $m_S(f) \leq m_{S_i}(f)$ since $S_i \subset S$ and thus

$$m_S(f)v(S) = m_S(f)v(S_1) + \cdots + m_S(f)v(S_\alpha) \leq m_{S_1}(f)v(S_1) + \cdots + m_{S_\alpha}(f)v(S_\alpha)$$

which yields

$$L(f,P) \le L(f,P')\,.$$

The proof of the statement about upper sums is analogous. □

Remark 7.1.3: Let P, P' be two partitions of a rectangle A. Then there exists a partition P'' of A which refines both P and P'.

Corollary 7.1.4. *If P and P' are any two partitions of the rectangle A. Then*

$$L(f,P) \le U(f,P')$$

Proof. Let P'' be a partition that refines both P and P'. Then

$$L(f,P) \le L(f,P'') \le U(f,P'') \le U(f,P)\,.$$

□

Definition 7.1.5. *Let $A \subset \mathbb{R}^n$ be a rectangle and $f : A \to \mathbb{R}$ and denote with $\mathcal{P}(A)$ the set of partitions of A. Denote*

$$\mathbf{L}\int_A f = \sup_{P\in\mathcal{P}(A)} L(f,P) \quad \text{and} \quad \mathbf{U}\int_A f = \inf_{P\in\mathcal{P}(A)} U(f,P)\,.$$

It follows from Corollary 7.1.4 that $\mathbf{L}\int_A f \le \mathbf{U}\int_A f$. We say f is integrable *on the rectangle A if f is bounded and*

$$\mathbf{L}\int_A f = \mathbf{U}\int_A f\,.$$

In this case, we denote

$$\int_A f = \mathbf{L}\int_A f = \mathbf{U}\int_A f\,,$$

which we call the integral *of f over A.*

Theorem 7.1.6 (Riemann's criterion)**.** *Let $A \subset \mathbb{R}^n$ be a rectangle and $f : A \to \mathbb{R}$ be bounded. Then f is integrable on A if and only if for all $\varepsilon > 0$ there exists $P \in \mathcal{P}(A)$ such that*

$$U(f,P) - L(f,P) < \varepsilon\,.$$

Proof. If he above criterion holds, then clearly

$$\sup_{P\in\mathcal{P}(A)} L(f,P) = \inf_{P\in\mathcal{P}(A)} U(f,P)$$

and f is integrable on A. On the other hand, if f is integrable on A then for any $\varepsilon > 0$ there exist $P, P' \in \mathcal{P}(A)$ such that

$$U(f,P) - L(f,P') < \varepsilon \,.$$

Let P'' be a partition that refines both P and P'. Then by Lemma 7.1.2

$$U(f,P'') - L(f,P'') \leq U(f,P) - L(f,P') < \varepsilon \;.$$

This proves the statement. □

Example 7.1.7: (1) Let $f : A \to \mathbb{R}$ be a constant function, i.e. $f(x) = c$ for some $c \in \mathbb{R}$ and for all $x \in A$. Then for any $P \in \mathcal{P}(A)$ and subrectangle S we have $m_S(f) = M_S(f) = c$ and thus $L(f,P) = U(f,P) = \sum_S cv(S) = cv(A)$. Thus f is integrable on A and $\int_A f = cv(A)$.

(2) Every continuous function $f : A \to \mathbb{R}$ is integrable, see assignment sheet 3.

(3) Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be such that

$$f(x,y) = \begin{cases} 0 & \text{if } x \text{ rational}\,, \\ 1 & \text{if } x \text{ irrational}\,. \end{cases}$$

Then for any P we have $U(f,P) = 1$ and $L(f,P) = 0$ and thus f is *not* integrable.

We collect some important properties of the integral.

Proposition 7.1.8. *Let $A \subset \mathbb{R}^n$ a closed rectangle and $f, g : A \to \mathbb{R}$. Then the following statements hold.*

(a) Assume f is integrable and $f = g$ except at finitely many points. Then g is integrable and $\int_A f = \int_A g$.

(b) Assume both f, g are integrable. Then $f + g$ is integrable and $\int_A f + g = \int_A f + \int_A g$.

(c) Assume f is integrable and $c \in \mathbb{R}$. Then cf is integrable and $\int_A cf = c \int_A f$.

(d) Assume f is integrable and let P be a partition of A. Then for each subrectangle $S \in P$ the restriciton of f to S, denoted with $f|_S$, is integrable on S. Furthermore $\int_A f = \sum_{S\in P} \int_S f|_S$.

(e) Assume both f, g are integrable and $f \leq g$. Then $\int f \leq \int g$.

(f) Assume f is integrable. Then $|f|$ is integrable and $\left|\int_A f\right| \leq \int_A |f|$.

Proof. See example sheet 5. □

7.2 Measure zero and integrable functions

Definition 7.2.1. *$A \subset \mathbb{R}^n$ has (n-dimensional)* measure *0 if for all $\varepsilon > 0$ there exists a cover $\{U_1, U_2, \ldots\}$ of A by closed rectangles such that $\sum_{i=1}^{\infty} v(U_i) < \varepsilon$.*

Remark 7.2.2: (1) We obviously assume as well that the rectangles U_i are open.

(2) Note that if A has measure 0 and $B \subset A$, then B has measure 0.

(3) Note that if A is countable then A has measure zero. To see this cover A by a sequence of rectangles U_i with $v(U_i) < 2^{-i}\varepsilon$. Then $\sum_{i=1}^{\infty} v(U_i) < \sum_{i=1}^{\infty} 2^{-i}\varepsilon = \varepsilon$.

Proposition 7.2.3. *If $A = A_1 \cup A_2 \cup A_3 \cup \cdots$ and each A_i has measure 0, then A has measure 0.*

Proof. Let $\varepsilon > 0$. Since A_i has measure 0, there is a cover $\{U_{i,1}, U_{i,2}, \ldots\}$ of A_i by closed rectangles such that $\sum_{j=1}^{\infty} v(U_{i,j}) < \varepsilon 2^{-i}$. Then the collection of all $U_{i,j}$ is a cover of A. We can now arrange this array into one sequence $V_1, V_2, V_3, \ldots$ (see picture in lectures). Clearly $\sum_{i=1}^{\infty} v(V_i) \leq \sum_{i=1}^{\infty} \varepsilon 2^{-i} = \varepsilon$. □

We have the following characterisation of bounded integrable functions. For the proof (which is neither difficult nor long) see [1, Theorem 3.8].

Theorem 7.2.4. *Let $A \subset \mathbb{R}^n$ be a closed rectangle and $f : A \to \mathbb{R}$ be bounded. Let $B := \{x \in A \mid f$ is not continuous at $x\}$. Then f is integrable if and only if B is a set of measure 0.*

Remark 7.2.5: Note that this implies that if $f, g : A \to \mathbb{R}$ are bounded and integrable on A, then $f \cdot g$ is integrable on A, see assignment sheet 3.

We have this far dealt only with integrals of functions over rectangles. Integrals of other sets are easily reduced to this type. For $C \subset \mathbb{R}^n$, the *characteristic function* χ_C of C is defined via

$$\chi_C(x) := \begin{cases} 0 & x \notin C\,, \\ 1 & x \in C\,. \end{cases}$$

If $C \subset A$ for some closed rectangle A and $f : A \to \mathbb{R}$ bounded, then $\int_C f$ is defined as $\int_A f \cdot \chi_C$, provided $f \cdot \chi_C$ is integrable. By the remark above this holds if f and χ_C are integrable.

We have the following characterisation when χ_C is integrable, for a proof see assignment sheet 3.

Theorem 7.2.6. *For $C \subset A$ for some closed rectangle A, the function χ_C is integrable if and only if the boundary of C has measure 0.*

7.3 Fubini's theorem

In this section we will discuss and prove Fubini's theorem, i.e. when it is possible to compute a higher dimensional integral by computing one-dimensional iterated integrals.

Idea: consider $f : [a, b] \times [c, d] \to \mathbb{R}$ positive and continuous. Let $t_0 < \cdots < t_k$ be a partition of

$[a,b]$ and divide $[a,b]\times[c,d]$ into k strips $[t_{i-1},t_i]\times[c,d]$ (compare the picture in the lectures). Let $g_x(y) := f(x,y)$, then the area under graph(f) above $\{x\}\times[c,d]$ is

$$\int_c^d g_x = \int_c^d f(x,y)\,dy\,.$$

So the volume under graph(f) above $[t_{i-1},t_i]\times[c,d]$ is therefore approximately equal to $(t_i - t_{i-1})\cdot \int_c^d f(x,y)\,dy$ for any $x\in[t_{i-1},t_i]$. Thus

$$\int_{[a,b]\times[c,d]} f = \sum_{i=1}^k \int_{[t_{i-1},t_i]\times[c,d]} f \approx \sum_{i=1}^k (t_i - t_{i-1})\cdot \int_c^d f(x_i,y)\,dy \approx \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx$$

for some choice of $x_i \in [t_{i-1},t_i]$.

Problem: f might be integrable on $[a,b]\times[c,d]$, but not continuous. Even more, $\int_c^d f(x_0,y)\,dy$ might not be defined for some $x_0\in[a,b]$. This will make the statement of Fubini's theorem a bit cumbersome, but we will see in the remarks that there are various special cases where simpler statements are possible.

Theorem 7.3.1 (Fubini's theorem)**.** *Let $A\subset\mathbb{R}^n, B\subset\mathbb{R}^m$ be closed rectangles and $f: A\times B\to\mathbb{R}$ be integrable. For $x\in A$ let $g_x: B\to\mathbb{R}, g_x(y) = f(x,y)$ and*

$$\mathcal{L}(x) = \mathbf{L}\int_B g_x = \mathbf{L}\int_B f(x,y)\,dy \qquad \mathcal{U}(x) = \mathbf{U}\int_B g_x = \mathbf{U}\int_B f(x,y)\,dy\,.$$

Then $\mathcal{L}$ and $\mathcal{U}$ are integrable on A and

$$\int_{A\times B} f = \int_A \mathcal{L} = \int_A \left(\mathbf{L}\int_B f(x,y)\,dy\right),$$
$$\int_{A\times B} f = \int_A \mathcal{U} = \int_A \left(\mathbf{U}\int_B f(x,y)\,dy\right).$$

The integrals on the RHS are called iterated integrals *of f.*

Proof. Let P_A be a partition of A and P_B a partition of B. This yields a partition of $A\times B$ with subrectangles of the form $S_A\times S_B$. Thus

$$L(f,P) = \sum_S m_S(f)v(S) = \sum_{S_A,S_B} m_{S_A\times S_B}(f)v(S_A\times S_B) = \sum_{S_A}\left(\sum_{S_B} m_{S_A\times S_B}(f)v(S_B)\right) v(S_A)\,.$$

If $x\in S_A$, then clearly $m_{S_A\times S_B}(f)\le m_{S_B}(g_x)$. Thus for $x\in S_A$

$$\sum_{S_B} m_{S_A\times S_B}(f)v(S_B) \le \sum_{S_B} m_{S_A\times S_B}(g_x)v(S_B) \le \mathbf{L}\int_B g_x = \mathcal{L}(x)\,.$$

Taking the infimum over S_A yields

$$\sum_{S_B} m_{S_A\times S_B}(f)v(S_B) \le m_{S_A}(\mathcal{L})$$

which implies

$$L(f,P) = \sum_{S_A}\left(\sum_{S_B} m_{S_A\times S_B}(f)v(S_B)\right)v(S_A) \le L(\mathcal{L},P_A)\,.$$

Similarly we have $U(\mathcal{U},P_A) \le U(f,P)$ and thus

$$L(f,P) \le L(\mathcal{L},P_A) \le U(\mathcal{L},P_A) \le U(\mathcal{U},P_A) \le U(f,P)\,.$$

Now note that any partition P of $A\times B$ can be written as a product of partitions P_A and P_B, and since f is integrable we have

$$\sup_P L(f,P) = \inf_P U(f,P) = \int_{A\times B} f\,.$$

This implies

$$\int_{A\times B} f = \int_A \mathcal{L}\,.$$

Similarly for $\mathcal{U}$ the statement follows from

$$L(f,P) \le L(\mathcal{L},P_A) \le L(\mathcal{U},P_A) \le U(\mathcal{U},P_A) \le U(f,P)\,.$$

□

Remark 7.3.2: (1) If for all $x\in A$ we have that $g_x(y) = f(x,y)$ is integrable on B then

$$\mathcal{L}(x) = \mathcal{U}(x) = \int_B f(x,y)\,dy$$

and

$$\int_{A\times B} f = \int_A\left(\int_B f(x,y)\,dy\right)dx\,. \tag{7.1}$$

This certainly occurs if $f : A\times B \to \mathbb{R}$ is continuous.

(2) Often g_x is not integrable for finitely many x. In this case we can still write

$$\mathcal{L}(x) = \int_B f(x,y)\,dy$$

for all but finitely many x. Since $\int_A \mathcal{L}$ remains unchanged if $\mathcal{L}$ is redefined at a finite number of points (say we set $\int_B f(x,y)\,dy = 0$ at these points) we still have (7.1).

(3) There are cases when this will not work and Theorem 7.3.1 must be used as stated. For example,

let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined via

$$f(x,y) := \begin{cases} 1 & \text{if } x \text{ is irrational,} \\ 1 & \text{if } x \text{ is rational and } y \text{ is irrational,} \\ 1 - \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms and } y \text{ is rational.} \end{cases}$$

Then f is integrable (exercise) and $\int_{[0,1]\times[0,1]} f = 1$. Now $\int_0^1 f(x,y)\,dy = 1$ if x is irrational, and does not exist if x is rational. Therefore $h(x) = \int_0^1 f(x,y)\,dy$ is not integrable even if we set $h(x) = 0$ when the integral does not exist.

(4) If $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and $f : A \to \mathbb{R}$ is sufficiently nice, we can apply Fubini's theorem repeatedly to obtain

$$\int_A f = \int_{a_n}^{b_n} \left(\cdots \left(\int_{a_1}^{b_1} f(x_1, \ldots, x_n)\, dx_1 \right) \cdots \right) dx_n \,.$$

Note that this also allows to interchange the order of integration, provided again that f is sufficiently nice (say continuous).

7.4 Partitions of unity

We introduce a highly important tool in the theory of integration. Recall that for $U \subset \mathbb{R}^n$ open, a function is in $C^\infty(U)$ if it is in $C^k(U)$ for every $k = 0, 1, 2, \ldots$. We record the following lemma.

Lemma 7.4.1. *(1) Let $U \subset \mathbb{R}^n$ be open and $C \subset U$ be compact. Then there is a compact set $D \subset U$ such that C is contained in the interior of D.*

(2) There is a function $f \in C^\infty(U)$ such that $f(U) \subset [0,1], f(x) = 1$ for all $x \in C$ and $f = 0$ outside some closed set contained in U.

Proof. See example sheet 5. □

Recall that for $A \subset \mathbb{R}^n$ a family of open sets $\mathcal{O}$ is called an *open cover* of A, provided

$$A \subset \bigcup_{O \in \mathcal{O}} O \,.$$

Theorem 7.4.2 (Existence of partition of unity)**.** *Let $A \subset \mathbb{R}^n$ and $\mathcal{O}$ be an open cover of A. Then there is a collection of C^∞ functions Φ, with each $\varphi \in \Phi$ defined on $U = \cup_{O \in \mathcal{O}} O$, with the following properties*

(1) For each $x \in U$ and $\varphi \in \Phi$ it holds $0 \le \varphi(x) \le 1$.

(2) *For each $x \in A$ there is an open neighborhood V_x of x such that all but finitely many $\varphi \in \Phi$ vanish on V_x.*

(3) *For each $x \in A$ it holds $\sum_{\varphi\in\Phi} \varphi(x) = 1$ (note that by (2) this sum is finite on V_x).*

(4) *For each $\varphi \in \Phi$ there is an open set $O \in \mathcal{O}$ such that $\varphi = 0$ outside some closed set contained in O.*

Remark 7.4.3: A collection Φ satisfying (1)-(3) is called a smooth *partition of unity* for A. If Φ also satisfies (4), it is said to be *subordinate* to the cover $\mathcal{O}$.

Proof. We divide the proof in several cases.

Case 1. *A is compact*

If A is compact, then a finite number of $O_1, \ldots, O_m$ of open sets in $\mathcal{O}$ cover A. It clearly suffices to construct a partition of unity subordinate to the cover $\{O_1, \ldots, O_m\}$. We will first find compact sets $D_i \subset O_i$ whose interiors cover A. The sets D_i are constructed inductively as follows. Suppose that $D_1, \ldots, D_k$ have been chosen so that $\{\operatorname{int} D_1, \ldots, \operatorname{int} D_k, O_{k+1}, \ldots, O_m\}$ covers A. Consider

$$C_{k+1} = A \setminus (\operatorname{int} D_1 \cup \ldots \cup \operatorname{int} D_k \cup O_{k+2} \cup \ldots \cup O_m)\,.$$

Then $C_{k+1} \subset U_{k+1}$ is compact. Hence by Lemma 7.4.1 (1) we can find a compact set D_{k+1} such that

$$C_{k+1} \subset \operatorname{int} D_{k+1} \quad \text{and} \quad D_{k+1} \subset O_{k+1}\,,$$

and thus $\{\operatorname{int} D_1, \ldots, \operatorname{int} D_{k+1}, O_{k+2}, \ldots, O_m\}$ covers A. This constructs the sets $D_1, \ldots, D_m$.

Having constructed the sets $D_1, \ldots, D_m$, by Lemma 7.4.1 (2), we can choose ψ_i to be a non-negative C^∞-function which is positive on D_i and 0 outside of some closed set contained in O_i. Since $\{D_1, \ldots, D_m\}$ covers A, we have $\psi_1(x) + \cdots + \psi_m(x) > 0$ for all x in some open set Ω containing A. On Ω we can define

$$\varphi_i(x) := \frac{\psi_i(x)}{\psi_1(x) + \cdots + \psi_m(x)}\,.$$

Again by by Lemma 7.4.1 (2) we can then choose $\eta : \Omega \to [0,1]$ a C^∞-function which is 1 on A and 0 outside a closed set contained in Ω. Then

$$\{\eta\varphi_i, \ldots, \eta\varphi_m\}$$

is the desired partition of unity.

Case 2. *$A = A_1 \cup A_2 \cup \cdots$, where each A_i is compact and $A_i \subset \operatorname{int} A_{i+1}$.*

For each i let $\mathcal{O}_i$ consist of all $O \cap (\operatorname{int} A_{i+1} \setminus A_{i-2})$ for $O \in \mathcal{O}$. Then $\mathcal{O}_i$ is an open cover of the compact set $B_i = A_i \setminus \operatorname{int} A_{i-1}$. By case 1 there is a partition of unity Φ_i for B_i subordinate to $\mathcal{O}_i$. For each $x \in A$, the sum

$$\sigma(x) = \sum_{\varphi\in\cup_i\Phi_i} \varphi(x)$$

is a finite sum in an open neighbourhood of x, since if $x \in A_i$ we have $\varphi(x) = 0$ for $\varphi \in \Phi_j$ for $j \geq i+2$. For each $\varphi \in \cup_i \Phi_i$ define $\tilde{\varphi}(x) = \varphi(x)/\sigma(x)$. The collection of all $\tilde{\varphi}$ is the desired partition of unity.

Case 3. *A is open.*

Let

$$A_i := \{x \in A \,|\, \|x\| \leq i \text{ and } \operatorname{dist}(x, \partial A) \geq 1/i\}$$

and apply case 2.

Case 4. *A is arbitrary.*

Let $U = \cup_{O \in \mathcal{O}} O$. By case 3 there is a partition of unity for U; this is also a partition of unity for A. □

Remark 7.4.4: (1) We note an important consequence of condition (2) in the above theorem. Let $C \subset A$ be compact. For each $x \in C$ there is an open set V_x containing x such that only finitely many $\varphi \in \Phi$ are not 0 on V_x. Since C is compact, finitely many such V_x cover C. Thus only finitely many $\varphi \in \Phi$ are not 0 on in an open neighborhood of C.

(2) We will see that an important application of partitions of unity will be to piece together results obtained only locally. For example we have so far only defined integrals over (suitable) subsets of closed rectangles. Partitions of unity can be used to define the integral of a function f over an open set $U \subset \mathbb{R}^n$ (even if U is unbounded). See [1, Theorem 3.12]. We will encounter several other applications later in the course.

7.5 Change of variables formula

Motivation. Assume $g : [a,b] \to \mathbb{R}$ is continuously differentiable and $f : \mathbb{R} \to \mathbb{R}$ is continuous. Then substitution implies that

$$\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) g' \,.$$

We can see this as follows. Let $F : \mathbb{R} \to \mathbb{R}$ be such that $F' = f$. By the chain rule

$$(F \circ g)' = (f \circ g) g'$$

and thus

$$\int_{g(a)}^{g(b)} f = F(g(b)) - F(g(a)) = (F \circ g)(b) - (F \circ g)(a) = \int_a^b (F \circ g)' = \int_a^b (f \circ g) g' \,.$$

If we assume that g is injective, and thus $g' \geq 0$ or $g' \leq 0$ on (a,b) we can write this as

$$\int_{g([a,b])} f = \int_{[a,b]} f \circ g \, |g'| \,.$$

The analog in higher dimensions is given in the following theorem.

Theorem 7.5.1 (Change of variables formula)**.** *Let $A \subset \mathbb{R}^n$ be open, $g : A \to \mathbb{R}^n$ be injective and continuously differentiable. If $f : g(A) \to \mathbb{R}$ is integrable, then*

$$\int_{g(A)} f = \int_A f \circ g \, |\det \partial g| \, .$$

For a proof see [1, Theorem 3.13 and following remarks]. The proof is not too difficult, but due to time constraints we have chosen not cover it in this course.

Remark 7.5.2: (1) In case g is an affine map, then $\mathrm{vol}(g(A)) = |\det(\partial g)|\mathrm{vol}(A)$ (see example sheet 5). This confirms the above formula if f is a constant function and g is affine.

(2) By approximating g locally by affine maps (i.e. using that g is assumed to be continuously differentiable) and (1) one can give a proof of the change of variables formula.

8 The divergence theorem

8.1 Integration on hypersurfaces

Consider $S \subset \mathbb{R}^n$. We say that S is a *smooth hypersurface* if it is an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. Recall from Section 5.1, especially Proposition 5.1.3 and Definition 5.1.4 that this means that equivalently either (i) or (ii) hold:

(i) For each $x \in S$ there exists $U \subset \mathbb{R}^n$ an open neighborhood of x and $f : U \to \mathbb{R}$ smooth such that $S \cap U = f^{-1}(0)$ and 0 is a regular value of f (i.e. $\nabla f(x) \neq 0$ for all $x \in S$) .

(ii) For each $x \in S$ there exists $U \subset \mathbb{R}^n$ an open neighborhood of x such that $S \cap U$ can be written as the graph of a smooth function $g : V \to \mathbb{R}$ where $V \subset \mathbb{R}^{n-1}$ is open (note that $\mathbb{R}^{n-1}$ corresponds here to the first $n-1$ coordinates, *after suitably relabelling the coordinates*).

Definition 8.1.1 (Smooth set). *An open set $\Omega \subset \mathbb{R}^n$ is called* smooth *if $\partial\Omega$ is a* smooth hypersurface *in $\mathbb{R}^n$. Note that this implies that for each point $x \in \partial\Omega$ there is $U \subset \mathbb{R}^n$ an open neighborhood of x such that $\partial\Omega \cap U$ can be written as the graph of a smooth function $g : V \to \mathbb{R}$ where $V \subset \mathbb{R}^{n-1}$ is open and $\Omega \cap U$ corresponds to the points* above *the graph of g.*

Remark 8.1.2: Assume $\Omega \subset \mathbb{R}^n$ is smooth and bounded. Then one can easily show that $\partial\Omega$ has measure zero (exercise). Thus by the results in Section 7.2 the characteristic function χ_Ω is integrable and we can integrate functions over Ω.

Motivation for the definition of the integral along a hypersurface. Recall that for $\gamma : [a, b] \to \mathbb{R}^n$ a regular curve (i.e. $\gamma'(t) \neq 0$ for all $t \in (a, b)$) the length of γ is defined as

$$l(\gamma) = \int_a^b \|\gamma'(t)\| \, dt$$

and for a continuous function $h : \gamma([a, b]) \to \mathbb{R}$ we can define its integral along γ via

$$\int_\gamma h \, ds := \int_a^b h(\gamma(t)) \, \|\gamma'(t)\| \, dt \, .$$

We can see that this does not depend on the paramatrisation of γ, i.e. let $\phi : [c, d] \to [a, b]$ be such that $\phi(c) = a$ and $\phi(d) = b$ and $\phi'(t) > 0$ for all $t \in [c, d]$. Consider

$$\tilde{\gamma} : [c, d] \to \mathbb{R}^n, t \mapsto (\gamma \circ \phi)(t) \, ,$$

which is called a *reparametrisation* of γ. Note that

$$\tilde{\gamma}' = (\gamma \circ \phi)' = \phi' \cdot (\gamma' \circ \phi)$$

and we can compute, using the change of variable formula (and that $\phi' > 0$)

$$\begin{aligned}\int_{\tilde{\gamma}} h\, ds &= \int_c^d h(\tilde{\gamma}(t))\, \|\tilde{\gamma}'(t)\|\, dt = \int_c^d h(\gamma(\phi(t))\, \|\phi'(t)\gamma'(\phi(t))\|\, dt \\ &= \int_c^d h(\gamma(\phi(t)))\, \|\gamma'(\phi(t))\|\, \phi'(t)dt = \int_a^b h(\gamma(t))\, \|\gamma'(t)\|\, dt = \int_\gamma h\, ds\end{aligned}$$

Note that if we consider a curve γ in the plane (i.e. in $\mathbb{R}^2$), which is given as the graph of a function $f : [a, b] \to \mathbb{R}$, i.e. $\gamma(t) = (t, f(t))$ we have that

$$\gamma'(t) = (1, f'(t)) \quad \text{and} \quad \|\gamma'(t)\| = \sqrt{1 + (f'(t))^2}\,,$$

and thus for $h : \gamma([a, b]) \to \mathbb{R}$ continuous

$$\int_\gamma h\, ds = \int_a^b h(t, g(t))\sqrt{1 + (g'(t))^2}\, dt\,.$$

We would like to generalise this to be able to integrate over a hypersurface which is given as the graph of a function $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^{n-1}$ is open. We assume first that f is affine, i.e. $f(\hat{x}) := c+L(\hat{x})$ where $c \in \mathbb{R}$ and $L \in L(\mathbb{R}^{n-1}, \mathbb{R})$. Note that this implies that there is $\hat{w} \in \mathbb{R}^{n-1}$ such that $L(\hat{v}) = \langle \hat{w}, \hat{v}\rangle$ for all $\hat{v} \in \mathbb{R}^{n-1}$. We then consider a parametrisation of $S = \text{graph}(f)$ over U, i.e. the map

$$F : U \to \mathbb{R}^n, F(\hat{x}_1, \ldots, \hat{x}_{n-1}) = \big(\hat{x}_1, \ldots, \hat{x}_{n-1}, L(\hat{x}_1, \ldots, \hat{x}_{n-1})\big).$$

We now would like to compute the $(n-1)$-dimensional volume of $F([0, 1] \times \cdots \times [0, 1])$. To do this note that for $\hat{x} \in U$

$$DF(\hat{x})(\hat{v}) =: \hat{L}(\hat{v}) = (\hat{v}, L(\hat{v})) = (\hat{v}, \langle \hat{w}, \hat{v}\rangle) \tag{8.1}$$

and

$$V := DF(\hat{x})(\mathbb{R}^{n-1}) = T_{F(\hat{x})}S$$

is the tangent space of S at $F(\hat{x})$, compare remark 5.1.5. Note that V is an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. From example sheet 5 we know that the $(n-1)$-dimensional volume of $F([0, 1] \times \cdots \times [0, 1])$ is given by the absolute value of the determinant of the linear map $\hat{L} : \mathbb{R}^{n-1} \to V$ (computed with respect to an orthonormal basis of $\mathbb{R}^{n-1}$ and an orthonormal basis of V). To compute this we can assume that $\hat{w} \neq 0$, otherwise there is not much to show. We now rotate the coordinate system in $\mathbb{R}^{n-1}$ such that $e_{n-1} = \hat{w}/\|\hat{w}\|$ (recall that the determinant of a map remains unchanged if one rotates a basis to another orthonormal basis). In these coordinates we have from (8.1) that

$$\hat{L}(\hat{v}_1, \ldots, \hat{v}_{n-1}) = (\hat{v}_1, \ldots, \hat{v}_{n-1}, \|\hat{w}\|\hat{v}_{n-1})$$

and thus an orthonormal basis of V is given by $\{e_1, \ldots, e_{n-2}, (1 + \|\hat{w}\|^2)^{-1/2}(e_{n-1} + \|w\|e_n)\}$. We furthermore see that with respect to these orthonormal bases

$$\det \hat{L} = \sqrt{1 + \|\hat{w}\|^2}\,. \tag{8.2}$$

Recall that for a general smooth function $f : U \to \mathbb{R}$ we have that its affine approximation at $a \in U$ is given by

$$\hat{x} \mapsto f(a) + Df(a)(\hat{x} - a) = f(a) - \langle \nabla f(a), a\rangle + \langle \nabla f(a), \hat{x}\rangle\,,$$

i.e. we can identify $\nabla f(a) = \hat{w}$ in (8.1). This motivates the following definition.

Definition 8.1.3 (Area of a graph)**.** *Let $U \subset \mathbb{R}^{n-1}$ be open and $f \in C^1(U)$. Then the* area A *of $S = graph(f) \subset \mathbb{R}^n$ is defined to be*

$$A(S) = \int_U \sqrt{1 + \|\nabla f\|^2} = \int_U \sqrt{1 + \|\nabla f(\hat{x})\|^2}\, d\hat{x}^{n-1}\,.$$

Furthermore for $h \in C^0(S)$ we define

$$\int_S h\, dA = \int_U h \circ F \sqrt{1 + \|\nabla f\|^2} = \int_U h((\hat{x}, f(\hat{x})))\sqrt{1 + \|\nabla f(\hat{x})\|^2}\, d\hat{x}^{n-1}\,,$$

where $F : U \to \mathbb{R}^n : \hat{x} \mapsto (\hat{x}, f(\hat{x}))$ and we write $d\hat{x}^{n-1}$ as a short form for $dx_1 \cdots dx_{n-1}$.

We now show that the above definition does not depend on in which direction we write S as a graph. We consider $\pi_i : \mathbb{R}^n \to \mathbb{R}^n$ the orthogonal projections parallel to the e_i-direction and denote $P_i = \operatorname{im} \pi_i$, i.e. the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ orthogonal to e_i. For $U \subset P_i$ open and $f : U \to \mathbb{R}$ we write

$$\operatorname{graph}_{P_i}(f) := \{\hat{x} + f(\hat{x})e_i \,|\, \hat{x} \in U\} \subset \mathbb{R}^n$$

for the graph of f over P_i.

Lemma 8.1.4. *For $i \in \{1, \ldots, n\}$, $f : U_i \to \mathbb{R}$ smooth and $U_i \subset P_i$ open, let $S = graph_{P_i}(f) \subset \mathbb{R}^n$. Assume that there exists $j \neq i$ and $g : U_j \to \mathbb{R}$ smooth with $U_j \subset P_j$ open and that we can also write $S = graph_{P_j}(g)$. Then for $h \in C^0(S)$ it holds*

$$\int_{U_i} h \circ F \sqrt{1 + \|\nabla_{P_i} f\|^2} = \int_{U_j} h \circ G \sqrt{1 + \|\nabla_{P_j} g\|^2}\,.$$

where we denote with $\nabla_{P_i}, \nabla_{P_j}$ the gradients on P_i, P_j, respectively.

Proof. Let $p \in S$. Since for an open neighborhood U of q there is $v \in C^\infty(U)$ such that $S \cap U = v^{-1}(0)$, where 0 is a regular value of v we have by remark 5.1.5 that $T_pS = \ker Dv(p)$ und thus T_pS does not depend on if we write $S = \operatorname{graph}_{P_i}(f)$ or $S = \operatorname{graph}_{P_j}(g)$.

Consider the maps $F : U_i \to \mathbb{R}^n, \hat{x} \mapsto \hat{x} + f(\hat{x})e_i$ and $G : U_j \to \mathbb{R}^n, \check{x} \mapsto \check{x} + g(\check{x})e_j$. Note that $F(U_i) = G(U_j) = S$ and π_i, π_j are the inverses of F and G, respectively. For $p = F(\hat{p}) = G(\check{p}) \in S$ note that

$$DF(\hat{p}) : \mathbb{R}^{n-1} \to V := T_pS \quad \text{and} \quad DG(\check{p}) : \mathbb{R}^{n-1} \to V\,,$$

where we consider $DF(\hat{p}), DG(\check{p})$ not as maps into $\mathbb{R}^n$ but into V. Thus by (8.2)

$$\det DF(\hat{p}) = \sqrt{1 + \|\nabla_{P_i} f(\hat{p})\|^2} \quad \text{and} \quad \det DG(\check{p}) = \sqrt{1 + \|\nabla_{P_j} g(\check{p})\|^2}\,.$$

Since $\pi_i\big|_V : V \to P_i$ is the inverse to $DF(\hat{p}) : \mathbb{R}^{n-1} \to V$ we obtain

$$\det \pi_i\big|_V(p) = \frac{1}{\sqrt{1 + \|\nabla_{P_i} f(\hat{p})\|^2}}\,.$$

Consider the map

$$\Psi : U_j \to U_i, \check{x} \mapsto (\pi_i \circ G)(\check{x})\,.$$

Note that Ψ is smooth, $\Psi(U_j) = U_i$ and $f \circ \Psi(\check{x}) = g(\check{x})$. Furthermore, we have by the above discussion

$$\det D\Psi(\check{x}) = \det D\pi_i(G(\check{x})) \cdot \det DG(\check{x}) = \frac{\sqrt{1 + \|\nabla_{P_j} g(\check{x})\|^2}}{\sqrt{1 + \|\nabla_{P_i} f(\hat{x})\|^2}}\,,$$

where $\hat{x} = \Psi(\check{x})$. Thus by the change of variable formula, Theorem 7.5.1 we have

$$\begin{aligned}
\int_{U_i} h \circ F \sqrt{1 + \|\nabla_{P_i} f\|^2} &= \int_{\Psi(U_j)} h \circ F \sqrt{1 + \|\nabla_{P_i} f\|^2} \\
&= \int_{U_j} h \circ F \circ \Psi \sqrt{1 + \|\nabla_{P_i} f\|^2}\, |\det D\Psi| \\
&= \int_{U_j} h \circ G \sqrt{1 + \|\nabla_{P_j} g\|^2}\,.
\end{aligned}$$

This yields the desired statement. □

We can now define the integral on the boundary of smooth, bounded open set. Note that $\partial\Omega$ is compact, so we can always assume that any open cover of $\partial\Omega$ is finite.

Definition 8.1.5. *Assume $\Omega \subset \mathbb{R}^n$ is open, bounded and smooth. Denote $S := \partial\Omega$ and let $\mathcal{O} = \{O_1, \ldots, O_N\}$ be a finite open cover of S such that $S \cap O_i$ can be written as a graph, with graphic parametrisation $F_i : U_i \to S \cap O_i$ where $U_i \subset \mathbb{R}^{n-1}$ is open and bounded, corresponding to $f_i : U_i \to \mathbb{R}$. Let Φ be a partition of unity of S subordinate to $\mathcal{O}$. By Remark 7.4.4 (1) we can assume that $\Phi = \{\varphi_1, \ldots, \varphi_M\}$ is finite. Since Φ is subordinate to $\mathcal{O}$ for each $\varphi_j \in \Phi$ we can choose $O_{i(j)} \in \mathcal{O}$ such that $\overline{\{x \in \mathbb{R}^n \,|\, \varphi_j(x) \neq 0\}} \subset O_{i(j)}$. Let $h \in C^0(S)$. We then define*

$$\int_S h\, dA = \sum_{j=1}^{M} \int_{U_{i(j)}} (\varphi_j h) \circ F_{i(j)} \sqrt{1 + \|\nabla f_{i(j)}\|^2}\,.$$

That this definition makes sense we need to check that it does not depend neither on the covering by graphical parametrisations nor on the choice of partition of unity.

Proposition 8.1.6. *The integral $\int_S h\, dA$ defined above does not depend on the choice of covering*

$\mathcal{O}$, the graphical parametrisations $F_i : U_i \to S \cap O_i$ and the partition of unity Φ.

Proof. Consider $\tilde{\mathcal{O}} = \{\tilde{O}_1, \dots, \tilde{O}_{\tilde{N}}\}$ a second open cover of S such that $S \cap \tilde{O}_l$ can be written as a graph, with graphic parametrisations $\tilde{F}_l : \tilde{U}_l \to S \cap \tilde{O}_l$, corresponding to $\tilde{f}_l : \tilde{U}_l \to \mathbb{R}$. Let $\tilde{\Phi} = \{\tilde{\varphi}_1, \dots, \tilde{\varphi}_{\tilde{M}}\}$ be a partition of unity of S subordinate to $\tilde{\mathcal{O}}$, where we similarly assign $\tilde{O}_{l(m)}$ to each $\tilde{\varphi}_m$. We can then write

$$\sum_{m=1}^{\tilde{M}} \int_{\tilde{U}_{l(m)}} (\tilde{\varphi}_m h) \circ \tilde{F}_{l(m)} \sqrt{1 + \|\nabla \tilde{f}_{l(m)}\|^2} = \sum_{j=1}^{M} \sum_{m=1}^{\tilde{M}} \int_{\tilde{U}_{l(m)}} (\varphi_j \tilde{\varphi}_m h) \circ \tilde{F}_{l(m)} \sqrt{1 + \|\nabla \tilde{f}_{l(m)}\|^2} \, .$$

Note that

$$\overline{\{x \in \mathbb{R}^n \mid \varphi_j(x)\tilde{\varphi}_m(x) \neq 0\}} \subset O_{i(j)} \cap \tilde{O}_{l(m)} \, .$$

So by Lemma 8.1.4 we have

$$\int_{\tilde{U}_{l(m)}} (\varphi_j \tilde{\varphi}_m h) \circ \tilde{F}_{l(m)} \sqrt{1 + \|\nabla \tilde{f}_{l(m)}\|^2} = \int_{U_{i(j)}} (\varphi_j \tilde{\varphi}_m h) \circ F_{i(j)} \sqrt{1 + \|\nabla f_{i(j)}\|^2} \, ,$$

and thus

$$\begin{aligned} \sum_{m=1}^{\tilde{M}} \int_{\tilde{U}_{l(m)}} (\tilde{\varphi}_m h) \circ \tilde{F}_{l(m)} \sqrt{1 + \|\nabla \tilde{f}_{l(m)}\|^2} &= \sum_{j=1}^{M} \sum_{m=1}^{\tilde{M}} \int_{\tilde{U}_{l(m)}} (\varphi_j \tilde{\varphi}_m h) \circ \tilde{F}_{l(m)} \sqrt{1 + \|\nabla \tilde{f}_{l(m)}\|^2} \\ &= \sum_{j=1}^{M} \sum_{m=1}^{\tilde{M}} \int_{U_{i(j)}} (\varphi_j \tilde{\varphi}_m h) \circ F_{i(j)} \sqrt{1 + \|\nabla f_{i(j)}\|^2} \\ &= \sum_{j=1}^{M} \int_{U_{i(j)}} (\varphi_j h) \circ F_{i(j)} \sqrt{1 + \|\nabla f_{i(j)}\|^2} \, . \end{aligned}$$

This gives the desired statement. □

8.2 Flux of a vectorfield

For $\Omega \subset \mathbb{R}^n$ open with smooth boundary, recall that for each $p \in \partial\Omega$ there is an open neighborhood U of p and $v \in C^\infty(U)$ such that $U \cap \partial\Omega = v^{-1}(0)$, where 0 is a regular value of v. Furthermore we can assume that $v < 0$ on $\Omega \cap U$ (otherwise replace v by $-v$). Recall that by Remark 5.1.5 we have that $T_p\partial\Omega = \ker Dv(p)$. Note that this implies that

$$T_p\partial\Omega = \{x \in \mathbb{R}^n \mid \langle \nabla v(p), x \rangle = 0\} \, .$$

Thus $\nabla v(p)$ is normal to $T_p\partial\Omega$ and since $v < 0$ on $\Omega \cap U$ we have that $\nabla v(p)$ points towards the outside of Ω. We can then take the *outward unit normal* to be

$$\nu(q) := \frac{\nabla v}{\|\nabla v\|}(q) \quad \text{for all } q \in \partial\Omega \cap U \, .$$

Note that since this choice is unique, and the above construction works in an open neighborhood of every point $p \in \partial\Omega$ we can extend this to a *global unit normal* along $\partial\Omega$.

Definition 8.2.1 (Outward unit normal)**.** *For $\Omega \subset \mathbb{R}^n$ open with smooth boundary we denote with $\nu : \partial\Omega \to \mathbb{R}^n$ the outward pointing unit normal.*

For the graph of a function there is nice formula for the unit normal along the graph of f.

Lemma 8.2.2. *Let $U \subset \mathbb{R}^{n-1}$ be open and $f \in C^1(U)$. Then the downward pointing unit normal of graph(f) at $(\hat{x}, f(\hat{x}))$ is given by*

$$\nu((\hat{x}, f(\hat{x})) = \frac{1}{\sqrt{1 + \|\nabla f\|^2(\hat{x})}}(\partial_1 f(\hat{x}), \ldots, \partial_{n-1} f(\hat{x}), -1)\,.$$

Here ∇ is the gradient on $\mathbb{R}^{n-1}$.

Proof. See assignment sheet 4. □

For $V \subset \mathbb{R}^n$ open, recall that $X \in C^0(V, \mathbb{R}^n)$ is also a called continuous vectorfield.

Definition 8.2.3 (Flux of a vectorfield)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary. Let $X \in C^0(\partial\Omega, \mathbb{R}^n)$ be a continuous vectorfield on $\partial\Omega$. The* flux *of X through $\partial\Omega$ is defined to be*

$$\int_{\partial\Omega} \langle X, \nu \rangle \, dA\,.$$

8.3 The divergence theorem

Definition 8.3.1. *Let $V \subset \mathbb{R}^n$ be open and $X \in C^1(V, \mathbb{R}^n)$ be a continuously differentiable vectorfield. Then* divergence *of $X = (X_1, \ldots, X_n)$ is defined to be*

$$\operatorname{div} X(x) = \sum_{i=1}^n \partial_i X_i(x)\,,$$

for all $x \in V$.

The next lemma covers the trivial case of the divergence theorem.

Proposition 8.3.2 (Divergence theorem: trivial case)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $X \in C^1(\Omega, \mathbb{R}^n)$ be such that X vanishes outside a closed set contained in Ω. Then*

$$\int_\Omega \operatorname{div} X = 0$$

Proof. Recall that since X vanishes outside a closed set contained in Ω we can extend X by the zero vectorfield to all of $\mathbb{R}^n$ and can treat X as an element of $X \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Let $R = R^{a,b}$ be any closed rectangle such that $\bar{\Omega} \subset \operatorname{int} R$. Note that since $X \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ we have that $\operatorname{div} X \in C^0(\mathbb{R}^n)$ and thus $\operatorname{div} X$ is integrable on R. Since X vanishes outside a closed set contained in Ω we have

$$\int_\Omega \operatorname{div} X = \int_R \chi_\Omega \operatorname{div} X = \int_R \operatorname{div} X .$$

Furthermore, by Fubini and the linearity of the integral

$$\int_R \operatorname{div} X = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \sum_{i=1}^n \partial_i X_i \, dx_n \ldots dx_1 = \sum_{i=1}^n \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \partial_i X_i \, dx_n \ldots dx_1 .$$

Note that for $i \in \{1, \ldots, n\}$ fixed we have by the fundamental theorem of calculus

$$\begin{aligned}\int_{a_i}^{b_i} \partial_i X_i(x_1, \ldots, x_n) dx_i &= X_i(x_1, \ldots, x_{i-1}, b_i, x_{i+1}, \ldots, x_n) \\ &\quad - X_i(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n) \\ &= 0 - 0 = 0 ,\end{aligned}$$

since X vanishes on ∂R. Since by Fubini we can interchange the order of integration we see that

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \partial_i X_i \, dx_n \ldots dx_1 = 0$$

and thus

$$\int_\Omega \operatorname{div} X = \int_R \operatorname{div} X = \sum_{i=1}^n \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \partial_i X_i \, dx_n \ldots dx_1 = 0 .$$

□

We now prove a localised version of the divergence theorem.

Proposition 8.3.3 (Divergence theorem: localised version)**.** *Let $V \subset \mathbb{R}^n$ open. Assume $R \subset V$ is a closed rectangle of the form $R' \times [a_n, b_n]$ where $R' \subset U$ is a ((n-1)-dimensional) rectangle and $U \subset \mathbb{R}^{n-1}$ is open. Consider $f \in C^1(U)$ such that $a_n < f(\hat{x}) < b_n$ for all $\hat{x} \in R'$ and let $S := graph(f|_{R'})$ and $\Omega = \{(\hat{x}, t) \in R \,|\, f(\hat{x}) < t < b_n\}$ the region above the graph of f in R. Let $X \in C^1(V, \mathbb{R}^n)$ be a vectorfield which vanishes on ∂R. Then*

$$\int_\Omega \operatorname{div} X = \int_S \langle X, \nu \rangle \, dA ,$$

where ν is the downward pointing unit normal of S.

Proof. To start, we claim

Claim. *If* $\varphi \in C^1(V)$ *and* $\varphi = 0$ *on* ∂R*, then for any* $j \in \{1, \dots, n\}$

$$\int_\Omega \partial_j \varphi = \int_S \varphi \nu_j \, dA \,. \tag{8.3}$$

Assuming the claim for now, and writing $X = (X_1, \dots, X_n)$ we see

$$\begin{aligned}\int_\Omega \operatorname{div} X &= \int_\Omega \sum_{j=1}^n \partial_j X_j = \sum_{j=1}^n \int_\Omega \partial_j X_j \\ &= \sum_{j=1}^n \int_S X_j \nu_j \, dA = \int_S \sum_{j=1}^n X_j \nu_j \, dA \\ &= \int_S \langle X, \nu \rangle \, dA \,,\end{aligned}$$

where we used the claim with $\varphi = X_j$ for each $j \in \{1, \dots, n\}$ from the first to the second line. This is the desired result.

Proof of claim.

Case $j = n$.

Note that since for $\hat{x} = (x_1, \dots, x_{n-1}) \in R'$ we have $\varphi(\hat{x}, b_n) = 0$, the fundamental theorem of calculus yields

$$-\varphi(\hat{x}, f(\hat{x})) = \varphi(\hat{x}, b_n) - \varphi(\hat{x}, f(\hat{x})) = \int_{f(\hat{x})}^{b_n} \partial_n \varphi(\hat{x}, x_n) \, dx_n \,. \tag{8.4}$$

We can thus compute, using Lemma 8.2.2, (8.4) and Fubini

$$\begin{aligned}\int_S \varphi \nu_n \, dA &= -\int_{R'} \varphi((\hat{x}, f(\hat{x}))) \frac{1}{\sqrt{1 + \|\nabla f(\hat{x})\|^2}} \sqrt{1 + \|\nabla f(\hat{x})\|^2} \, d\hat{x}^{n-1} \\ &= \int_{a_1}^{b_1} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{f((x_1, \dots, x_{n-1}))}^{b_n} \partial_n \varphi(x_1, \dots, x_n) \, dx_n \, dx_{n-1} \cdots dx_1 \\ &= \int_\Omega \partial_n \varphi \,.\end{aligned}$$

This yields (8.3) for $j = n$.

Case $j = 1, \dots, n-1$.

Note that by Leibnitz' rule (i.e. differentiating under the integral sign, see question 4 (a) on assignment sheet 3) and the chain rule, we have

$$\frac{\partial}{\partial x_j} \left(\int_{f(\hat{x})}^{b_n} \varphi((\hat{x}, x_n)) \, dx_n \right) = -\varphi((\hat{x}, f(\hat{x}))) \partial_j f(\hat{x}) + \int_{f(\hat{x})}^{b_n} \partial_j \varphi((\hat{x}, x_n)) \, dx_n \,,$$

and thus, since φ vanishes on ∂R

$$\begin{aligned}\int_{a_j}^{b_j}\int_{f(\hat{x})}^{b_n} \partial_j\varphi((\hat{x},x_n))\,dx_n\,dx_j &= \int_{a_j}^{b_j} \varphi((\hat{x},f(\hat{x})))\partial_j f(\hat{x})\,dx_j \\ &\quad + \int_{f(x_1,\dots,x_{j-1},b_j,x_{j+1},\dots,x_{n-1})}^{b_n} \varphi(((x_1,\dots,x_{j-1},b_j,x_{j+1},\dots,x_n))\,dx_n \\ &\quad - \int_{f(x_1,\dots,x_{j-1},a_j,x_{j+1},\dots,x_{n-1})}^{b_n} \varphi(((x_1,\dots,x_{j-1},a_j,x_{j+1},\dots,x_n))\,dx_n \\ &= \int_{a_j}^{b_j} \varphi((\hat{x},f(\hat{x})))\partial_j f(\hat{x})\,dx_j + 0 - 0 \\ &= \int_{a_j}^{b_j} \varphi((\hat{x},f(\hat{x})))\partial_j f(\hat{x})\,dx_j\,.\end{aligned} \tag{8.5}$$

We can then compute (where use the notation that a ' $\widehat{\ }$ ' over a symbol means that it is omitted), using Fubini, (8.5) and Lemma 8.2.2

$$\begin{aligned}\int_\Omega \partial_j\varphi &= \int_{a_1}^{b_1}\cdots\int_{a_{n-1}}^{b_{n-1}}\int_{f(\hat{x})}^{b_n} \partial_j\varphi((\hat{x},x_n)dx_n\,dx_{n-1}\cdots dx_1 \\ &= \int_{a_1}^{b_1}\cdots\widehat{\int_{a_j}^{b_j}}\cdots\int_{a_{n-1}}^{b_{n-1}}\int_{a_j}^{b_j}\int_{f(\hat{x})}^{b_n} \partial_j\varphi((\hat{x},x_n)dx_n\,dx_j\,dx_{n-1}\cdots\widehat{dx_j}\cdots dx_1 \\ &= \int_{a_1}^{b_1}\cdots\widehat{\int_{a_j}^{b_j}}\cdots\int_{a_{n-1}}^{b_{n-1}}\int_{a_j}^{b_j} \varphi((\hat{x},f(\hat{x})))\partial_j f(\hat{x})\,dx_j\,dx_{n-1}\cdots\widehat{dx_j}\cdots dx_1 \\ &= \int_{a_1}^{b_1}\cdots\int_{a_{n-1}}^{b_{n-1}} \varphi((\hat{x},f(\hat{x})))\partial_j f(\hat{x})\,dx_{n-1}\cdots dx_1 \\ &= \int_{R'} \varphi((\hat{x},f(\hat{x})))\frac{\partial_j f(\hat{x})}{\sqrt{1+\|\nabla f(\hat{x})\|^2}}\sqrt{1+\|\nabla f(\hat{x})\|^2}d\hat{x}^{n-1} \\ &= \int_S \varphi\nu_j\,dA\,.\end{aligned}$$

This yields (8.3) for $j = 1,\dots,n-1$. □

□

We are now in a position to collect the previous results to give a proof of the divergence theorem.

Theorem 8.3.4 (Divergence theorem)**.** *Let $V \subset \mathbb{R}^n$ be open and Ω be open, bounded with smooth boundary such that $\overline{\Omega} \subset V$. For any vectorfield $X \in C^1(V,\mathbb{R}^n)$ it holds*

$$\int_\Omega \operatorname{div} X = \int_{\partial\Omega} \langle X,\nu\rangle\,dA\,,$$

where ν is the outward unit normal vector to $\partial\Omega$.

Proof. Since $\partial\Omega$ is smooth, we can find for any $p \in \partial\Omega$ a rectangle $R_p \subset V$, centered at p, such that after relabeling the coordinates in $\mathbb{R}^n$ and/or switching $x_n \mapsto -x_n$ we have that $\partial\Omega \cap R_p$ is the graph of $f_p : R'_p \to ((a_p)_n, (b_p)_n)$, where $R_p = R'_p \times [(a_p)_n, (b_p)_n]$, and $\Omega \cap R_p$ is the region above $\operatorname{graph}(f_p)$. Replacing R_p with a slightly smaller rectangle (which we denote again by R_p) we see that we are in the setup of Proposition 8.3.3.

Note that $\{\operatorname{int} R_p\}_{p\in\partial\Omega}$ is an open cover of $\partial\Omega$. Since $\partial\Omega$ is compact, there exists a finite subcover $\mathcal{O} = \{\operatorname{int} R_{p_1}, \dots, \operatorname{int} R_{p_N}\}$. By Theorem 7.4.2 and Remark 7.4.4 (1) there is a finite partition of unity $\Phi = \{\varphi_1, \dots, \varphi_M\}$ of $\partial\Omega$ subordinate to $\mathcal{O}$.

This yields that for $l \in \{1, \dots, M\}$ there exists $k \in \{1, \dots, N\}$ such that the vectorfield $X_l := \varphi_l \cdot X$ satisfies

$$\{x \in V \mid X_l(x) \neq 0\} \subset R_{p_k} \ .$$

Note that this implies that X_l vanishes on ∂R_{p_k}. Thus we can apply Proposition 8.3.3 to obtain

$$\int_\Omega \operatorname{div} X_l = \int_{\partial\Omega} \langle X_l, \nu\rangle \, dA \tag{8.6}$$

for all $l \in \{1, \dots, M\}$.

Now consider the function

$$\eta : \Omega \to [0,1], x \mapsto 1 - \sum_{l=1}^M \varphi_l(x) \ .$$

Note that $\eta \in C^\infty(\Omega)$ and η vanishes outside a closed set contained in Ω. Thus we can extend η by zero to all of $\mathbb{R}^n$ and treat η as an element of $C^\infty(\mathbb{R}^n)$. Furthermore in an open neighborhood $V' \subset V$ of $\overline{\Omega}$ we have

$$\sum_{l=1}^M \varphi_l + \eta = 1 \ ,$$

and we can write

$$X = \sum_{l=1}^M X_l + \eta X \quad \text{on } V'$$

Since the vectorfield ηX vanishes outside a closed set contained in Ω we have by Proposition 8.3.2 that

$$\int_\Omega \operatorname{div}(\eta X) = 0 \, . \tag{8.7}$$

We can thus combine (8.6) and (8.7) to see that

$$\begin{aligned}\int_\Omega \operatorname{div} X &= \int_\Omega \operatorname{div}\left(\sum_{l=1}^M X_l + \eta \cdot X\right) = \int_\Omega \left(\sum_{l=1}^M \operatorname{div} X_l + \operatorname{div}(\eta X)\right)\\ &= \sum_{l=1}^M \int_\Omega \operatorname{div} X_l + \int_\Omega \operatorname{div}(\eta X) = \sum_{l=1}^M \int_{\partial\Omega} \langle X_l, \nu\rangle \, dA + 0\\ &= \int_{\partial\Omega} \langle X, \nu\rangle \, dA\,,\end{aligned}$$

since $X = \sum_{l=1}^M X_l$ on $\partial\Omega$.

This completes the proof of the divergence theorem. □

Remark 8.3.5: (1) *Physical interpretation:* In words, if we see the vectorfield X as the *speed vectorfield* of some quantity, then the divergence theorem asserts that the average rate at which the quantity flows out of Ω (i.e. the flux of X across $\partial\Omega$) is equal to the integral of $\operatorname{div} X$ over Ω. Now the vectorfield must, in a loosed sense, *diverge*, to have a net flow across $\partial\Omega$. This provides some justification for the terminology *divergence*.

(2) Note that if $n = 1$ we can take $\Omega = (a, b)$ for $a, b \in \mathbb{R}$. For e_1 the unit basis vector in $\mathbb{R}$ we see that the outward unit normal to $\partial\Omega$ is given by $\nu(a) = -e_1$ and $\nu(b) = e_1$. Furthermore for $[a, b] \subset (c, d)$ and a vectorfield $X \in C^1((c, d), \mathbb{R})$, we can write for $x \in (c, d)$

$$X(x) = f(x)e_1$$

for $f \in C^1((c, d))$. The divergence theorem then implies that

$$\int_{(a,b)} \operatorname{div} X = \langle X(a), \nu(a)\rangle + \langle X(b), \nu(b)\rangle = f(a)\langle e_1, -e_1\rangle + f(b)\langle e_1, e_1\rangle = f(b) - f(a)\,.$$

But we also have

$$\operatorname{div}(X)(x) = \partial_1 f(x) = f'(x)$$

and thus

$$\int_{(a,b)} \operatorname{div} X = \int_a^b f'(x)\, dx\,.$$

This yields that for $n = 1$ the divergence theorem is equivalent to the fundamental theorem of calculus:

$$\int_a^b f'(x)\, dx = f(b) - f(a)\,.$$

(3) Coming back to the physical interpretation, now for $n = 1$, we have seen that a function $f\colon [a, b] \to \mathbb{R}$ can be thought of as a vector field on the line. For instance, if we think of $[a, b]$ as a highway, then $f(x)$ could represent the number of vehicles per minute that pass by x; we are assuming that this rate is a function of position x but not a function of time which, of course, is unrealistic. Anyway, $f'(x)$ is then the divergence of f and $f(b) - f(a)$ is the flux of f across the

boundary $\partial(a,b)$, which consists of just the two points $\{a,b\}$. However, $f(b)$ represents the rate at which traffic is exiting the highway at b and $f(a)$ represents the rate at which traffic is entering the highway at a. Thus the flux $f(b)-f(a)$ represents the rate at which traffic is building up on the stretch $[a,b]$ of highway between a and b. The fundamental theorem of calculus (or equivalently the divergence theorem) states that this flux is equal to the integral of the divergence f' of the rate of traffic flow f along $[a,b]$.

8.3.1 The divergence theorem in the plane

Consider a bounded open set $\Omega \subset \mathbb{R}^2$ with smooth boundary. Note that in this case $\partial\Omega$ is compact, and it can be parametrised by a collection $\{\gamma_1,\dots,\gamma_K\}$ of disjoint, closed, embedded (i.e. no self-intersections) and regular curves. Here closed means that $\gamma_i : [a_i,b_i] \to \mathbb{R}^2$ and

$$\gamma^{(k)}(a_i) = \gamma^{(k)}(b_i)$$

for all $k \in \mathbb{N}\cup\{0\}$, in the sense of right and left derivatives. We will say that $\{\gamma_1,\dots,\gamma_K\}$ *parametrise* $\partial\Omega$.

Let us for the moment assume that $\partial\Omega$ has only one component, parametrised (with the conditions as above) by $\gamma : [a,b] \to \mathbb{R}^2$. Recall that in the definition of the integral along the boundary $\partial\Omega$, Definition 8.1.5, we considered $\mathcal{O} = \{O_1,\dots,O_N\}$ be a finite open cover of $\partial\Omega$ such that $\partial\Omega \cap O_i$ can be written as a graph, with graphic parametrisation $F_i : (a_i,b_i) \to \partial\Omega\cap O_i$, corresponding to $f_i : (a_i,b_i) \to \mathbb{R}$. Considering $\Phi = \{\varphi_1,\dots,\varphi_M\}$ a finite partition of unity of $\partial\Omega$ subordinate to $\mathcal{O}$ for each $\varphi_j \in \Phi$ we choose $O_{i(j)} \in \mathcal{O}$ such that $\overline{\{x\in\mathbb{R}^2 \,|\, \varphi_j(x)\neq 0\}} \subset O_{i(j)}$. For $h \in C^0(\partial\Omega)$ we then defined

$$\int_{\partial\Omega} h\,ds = \sum_{j=1}^{M} \int_{(a_{i(j)},b_{i(j)})} (\varphi_j h)\circ F_{i(j)} \sqrt{1+|f'_{i(j)}|^2}\,.$$

But in our case $\gamma : [a,b] \to \mathbb{R}^2$ is a *global* parametrisation of $\partial\Omega$ and we have by the invariance of the integral along a curve under reparametrisation (see the beginning of Chapter 8) that for each $j \in \{1,\dots,M\}$ we have

$$\int_{(a_{i(j)},b_{i(j)})} (\varphi_j h)\circ F_{i(j)} \sqrt{1+|f'_{i(j)}|^2} = \int_{[a,b]} (\varphi_j h)\circ\gamma\,\|\gamma'\| = \int_\gamma \varphi_j h\,ds\,.$$

This implies

$$\int_{\partial\Omega} h\,ds = \sum_{j=1}^{M} \int_{(a_{i(j)},b_{i(j)})} (\varphi_j h)\circ F_{i(j)} \sqrt{1+|f'_{i(j)}|^2} = \sum_{j=1}^{M} \int_\gamma \varphi_j h\,ds = \int_\gamma \sum_{j=1}^{M} \varphi_j h\,ds = \int_\gamma h\,ds\,,$$

and thus

$$\int_{\partial\Omega} h\,ds = \int_\gamma h\,ds = \int_a^b h\circ\gamma\,\|\gamma'\|\,dt\,,$$

i.e. both our definitions of the integral along the boundary (for $n=1$) and along a curve agree! (As they should for both definitions to be reasonable).

We can use the above discussion to restate the Divergence Theorem in the plane.

Theorem 8.3.6 (Divergence Theorem in the Plane)**.** *Let $V \subset \mathbb{R}^2$ be open and Ω be open, bounded with smooth boundary such that $\overline{\Omega} \subset V$. Assume $\{\gamma_1, \dots, \gamma_K\}$ are disjoint, closed, embedded, regular curves which parametrise $\partial\Omega$. Then for any vectorfield $X \in C^1(V, \mathbb{R}^2)$ it holds*

$$\int_\Omega \operatorname{div} X = \sum_{i=1}^K \int_{\gamma_i} \langle X, \nu \rangle \, ds \,,$$

where ν is the outward unit normal vector to $\partial\Omega$.

Remark 8.3.7: Note that if $\gamma_i : [a_i, b_i] \to \mathbb{R}^2$ is oriented in such a way that Ω lies locally *to the left* of $t \mapsto \gamma_i(t)$ and $\gamma' = (\gamma_1', \gamma_2')$, then we can write

$$\nu(\gamma(t)) = \frac{1}{\|\gamma'(t)\|} R\gamma'(t) = \frac{1}{\|\gamma'(t)\|} \begin{pmatrix} \gamma_2'(t) \\ -\gamma_1'(t) \end{pmatrix}$$

where $R : \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation by $\pi/2$ in clockwise direction. (If Ω lies to the left of $t \mapsto \gamma_i(t)$ we have to multiply the expression on the RHS by -1). We can thus write

$$\int_{\gamma_i} \langle X, \nu \rangle \, ds = \int_{a_i}^{b_i} \langle X(\gamma(t)), \nu(\gamma(t)) \rangle \|\gamma'(t)\| \, dt = \int_{a_i}^{b_i} \langle X(\gamma(t)), R\gamma'(t) \rangle \, dt \,.$$

Example 8.3.8: Consider the vectorfield $W(x, y) := (x, y)$ on $\mathbb{R}^2$ and the vectorfield $V(x, y) := \frac{1}{x^2+y^2}(x, y)$ on $\mathbb{R}^2 \setminus \{(0, 0)\}$. Calculate

(a) div W and div V,

(b) the flux of V across the circle C_1 of radius 1 with centre at $(0, 0)$.

Explain why the answer in (b) does not contradict the divergence theorem and the answer in (a) for V. Calculate

(c) the flux of V across the circle C_2 of radius 2 with centre at $(0, 0)$.

Explain why the answers in (b) and (c) are equal.

Solution.

(a) div $W = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2$.

$$\begin{aligned} \text{div } V &= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) \\ &= \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2} \\ &= 0. \end{aligned}$$

(b) If $(x, y) \in C_1$ then $\|(x, y)\| = 1$ and therefore, the unit normal $\nu(x, y)$ at $(x, y) \in C_1$ is equal to (x, y). It follows that

$$\int_{C_1} \langle V, \nu \rangle \, ds = \int_{C_1} \langle (x, y), (x, y) \rangle \, ds = \int_{C_1} 1 \, ds = \text{length}(C_1) = 2\pi.$$

It is not possible to define V at $(0, 0)$ in such a way as to make V continuous (or even differentiable) there. Therefore, the divergence theorem cannot be applied to V on the unit disk centred at the origin.

(c) If $(x, y) \in C_2$ then $\|(x, y)\| = 2$ and therefore, the unit normal $\nu(x, y)$ at $(x, y) \in C_2$ is equal to $\frac{1}{2}(x, y)$. It follows that

$$\int_{C_2} \langle V, \nu \rangle \, ds = \int_{C_2} \tfrac{1}{4}\langle (x, y), \left(\tfrac{1}{2}(x, y)\right)\rangle \, ds = \int_{C_2} \tfrac{1}{2} \, ds = \tfrac{1}{2}\,\text{length}(C_2) = 2\pi.$$

We may apply the divergence theorem to V on the annulus $A := \{(x, y) \in \mathbb{R}^2 \mid 1 \le x^2 + y^2 \le 4\}$. Then $\partial A = C_1 \cup C_2$. On C_2, the outer unit normal ν to A is equal to $\frac{1}{2}(x, y) = \nu(x, y)$, as in (c). However, on C_1, the outer unit normal ν is equal to $-(x, y) = -\nu(x, y)$, i.e. the opposite of the unit normal in (b). Therefore,

$$\int_{\partial A} \langle V, \nu \rangle \, ds = \int_{C_2} \langle V, \nu \rangle \, ds - \int_{C_1} \langle V, \nu \rangle \, ds = 0,$$

i.e., $\displaystyle\int_{\partial A} \langle V, \nu \rangle \, ds = \int_A \text{div } V \, dx \, dy$, in agreement with the divergence theorem.

8.3.2 Integration by parts and Green's formulas

We first note the following little formula. Consider $U \subset \mathbb{R}^n$ open and $X \in C^1(U, \mathbb{R}^n), \varphi \in C^1(U)$. We then have

$$\text{div}(\varphi X) = \sum_{i=1}^n \partial_i(\varphi X_i) = \varphi \sum_{i=1}^n \partial_i X_i + \sum_{i=1}^n (\partial_i \varphi) X_i = \varphi \text{ div } X + \langle \nabla \varphi, X \rangle \,. \tag{8.8}$$

Using the divergence theorem we can generalise integration by parts to higher dimensions - without appealing to Fubini.

Proposition 8.3.9 (Integration by parts)**.** *Let $V \subset \mathbb{R}^n$ be open and $\Omega \subset \mathbb{R}^n$ be bounded, open with smooth boundary and $\bar{\Omega} \subset V$. Let $u, v \in C^1(V)$ and $i \in \{1, \dots, n\}$, then*

$$\int_\Omega u\, \partial_i v = -\int_\Omega \partial_i u\, v + \int_{\partial\Omega} uv\, \nu_i\, dA$$

Proof. Consider e_i the i-th basis vector in $\mathbb{R}^n$. Note that we can write

$$\partial_i v = \operatorname{div}(v \cdot e_i)\,,$$

and thus, together with (8.8)

$$u\, \partial_i v = u \operatorname{div}(v \cdot e_i) = \operatorname{div}(uv \cdot e_i) - \langle \nabla u, v \cdot e_i \rangle = \operatorname{div}(uv \cdot e_i) - v\, \partial_i u\,.$$

Thus by the divergence theorem

$$\begin{aligned} \int_\Omega u\, \partial_i v = \int_\Omega \operatorname{div}(uv \cdot e_i) - v\, \partial_i u &= -\int_\Omega \partial_i u\, v + \int_{\partial\Omega} \langle uv \cdot e_i, \nu \rangle\, dA \\ &= -\int_\Omega \partial_i u\, v + \int_{\partial\Omega} uv\, \nu_i\, dA\,. \end{aligned}$$

□

Remark 8.3.10: (1) Note that if either u or v vanish on $\partial\Omega$, we obtain

$$\int_\Omega u\, \partial_i v = -\int_\Omega \partial_i u\, v\,. \tag{8.9}$$

(2) Even without any regularity assumptions on Ω (i.e. no smooth boundary), if say $u, v \in C^1(\Omega)$ and u vanishes outside a compact set contained in Ω, then by Proposition 8.3.2 we still have that (8.9) holds.

Definition 8.3.11 (Laplacian)**.** *Let $U \subset \mathbb{R}^n$ be open and $u \in C^2(U)$. We define the* Laplacian *of u at $x \in U$ to be*

$$\Delta u(x) := \sum_{i=1}^n \partial_i^2 u(x)\,.$$

We note that we also alternatively write for $x \in U$

$$\Delta u(x) = \sum_{i=1}^n D^2 u(e_i, e_i)(x) = \operatorname{tr} \partial^2 u(x) = \operatorname{div}(\nabla u)(x)\,, \tag{8.10}$$

where $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$.

Proposition 8.3.12 (Green's formulas)**.** *Let $V \subset \mathbb{R}^n$ be open and $\Omega \subset \mathbb{R}^n$ be bounded, open with smooth boundary and $\bar{\Omega} \subset V$. Assume $u, v \in C^2(V)$. Then*

$$\int_\Omega (u\,\Delta v + \langle \nabla u, \nabla v\rangle) = \int_{\partial\Omega} u\,\langle \nabla v, \nu\rangle\, dA\,, \tag{8.11}$$

and

$$\int_\Omega (u\,\Delta v - v\,\Delta u) = \int_{\partial\Omega} (u\,\langle \nabla v, \nu\rangle - v\,\langle \nabla u, \nu\rangle)\, dA\,, \tag{8.12}$$

Proof. Using (8.10) and (8.8) we can compute

$$u\,\Delta v = u\ \mathrm{div}(\nabla v) = \mathrm{div}(u\nabla v) - \langle \nabla u, \nabla v\rangle\,. \tag{8.13}$$

Integrating this over Ω and using the divergence theorem yields (8.11). Interchanging u and v in (8.13) and substracting from (8.13) yields

$$u\,\Delta v - v\,\Delta u = \mathrm{div}(u\nabla v) - \mathrm{div}(v\nabla u)\,.$$

Again integrating this over Ω and using the divergence theorem yields (8.12). □

8.3.3 Tangential line integral and Green's theorem in the plane

We recall the definition a the unit tangent vector to a regular curve.

Definition 8.3.13 (Unit tangent vector)**.** *Let $\gamma : [a, b] \to \mathbb{R}^n$ be a regular curve. Then its* unit tangent vector *$\tau(t)$ is defined to be*

$$\tau(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|}\,.$$

We can use this to define the tangential line integral of a vectorfield along a regular curve.

Definition 8.3.14 (Tangential line integral)**.** *Let $U \subset \mathbb{R}^n$ be open and $X \in C^0(U, \mathbb{R}^n)$. For a regular curve $\gamma : [a, b] \to U$ we define the* tangential line integral *of X along γ to be*

$$\int_\gamma \langle X, \tau\rangle\, ds := \int_\gamma \langle X \circ \gamma, \tau\rangle\, ds = \int_a^b \langle X(\gamma(t)), \tau(t)\rangle \|\gamma'(t)\|\, dt = \int_a^b \langle X(\gamma(t)), \gamma'(t)\rangle\, dt\,.$$

We should first check that this definition is invariant under a reparametrisation of γ. Recall that for a reparametrisation of γ we consider a continuously differentiable function $\phi : [c, d] \to [a, b]$ such that $\phi(c) = a$ and $\phi(d) = b$ and $\phi'(t) > 0$ for all $t \in [c, d]$. Then

$$\tilde{\gamma} : [c, d] \to \mathbb{R}^n, t \mapsto (\gamma \circ \phi)(t)\,,$$

is a *reparametrisation* of γ. Note that

$$\tilde{\gamma}' = (\gamma \circ \phi)' = \phi' \cdot (\gamma' \circ \phi)$$

and we can compute, using the change of variable formula

$$\begin{aligned}\int_{\tilde{\gamma}} \langle X \circ \tilde{\gamma}, \tilde{\tau} \rangle \, ds &= \int_c^d \langle X(\tilde{\gamma}(t)), \tilde{\gamma}'(t) \rangle \, dt = \int_{\phi(a)}^{\phi(b)} \langle X(\gamma(\phi(t))), \gamma'(\varphi(t)) \rangle \, \varphi'(t) dt \\ &= \int_a^b \langle X(\gamma(t)), \gamma'(t) \rangle \, dt = \int_\gamma \langle X \circ \gamma, \tau \rangle \, ds \,,\end{aligned}$$

and thus indeed the definition is invariant under reparametrisation.

Remark 8.3.15 (Physical interpretation)**:** When the vector field X represents a force, then $\int_\gamma \langle X, \tau \rangle \, ds$ represents the *work* done by X while pushing an object along γ.

When γ is a closed curve, $\int_\gamma \langle X, \tau \rangle \, ds$ is sometimes referred to as the *circulation* of X around γ as it measures the rate at which the quantity described by the vector field X circulates around γ. The symbol $\oint_\gamma$ is sometimes used to indicate the fact that γ is closed.

Finally we would like to discuss Green's theorem in the plane.

Definition 8.3.16 (Curl of a vectorfield)**.** *Let $U \subset \mathbb{R}^2$ be open and $X \in C^1(U, \mathbb{R}^2)$ then we define*

$$\operatorname{curl} X := \partial_1 X_2 - \partial_2 X_1 \,.$$

Remark 8.3.17: Note that if $R : \mathbb{R}^2 \to \mathbb{R}^2$ again denotes the clockwise rotation by $\pi/2$ then

$$R(X) = R((X_1, X_2)) = (X_2, -X_1)$$

and we can write

$$\operatorname{curl} X = \operatorname{div} R(X) \,.$$

Theorem 8.3.18 (Green's theorem in the plane)**.** *Let $V \subset \mathbb{R}^2$ be open and Ω be open, bounded with smooth boundary such that $\overline{\Omega} \subset V$. Assume $\{\gamma_1, \ldots, \gamma_K\}$ are disjoint, closed, embedded, regular curves which parametrise $\partial\Omega$, which are oriented in such a way such that Ω locally lies to the left of γ_i for $i = 1, \ldots, K$. Then for any vectorfield $X \in C^1(V, \mathbb{R}^2)$ it holds*

$$\int_\Omega \operatorname{curl} X = \sum_{i=1}^K \oint_{\gamma_i} \langle X, \tau \rangle \, ds \,,$$

where τ is the unit tangent vector along each γ_i.

Proof. From Remark 8.3.17 we have that

$$\operatorname{curl} X = \operatorname{div} R(X)\,,$$

and by the divergence theorem in the plane, Theorem 8.3.6

$$\int_\Omega \operatorname{curl} X = \int_\Omega \operatorname{div} R(X) = \sum_{i=1}^{K} \int_{\gamma_i} \langle R(X), \nu \rangle \, ds$$

where ν is the unit outward normal to $\partial\Omega$. Note that since R preserves the scalar product between two vectors (i.e. $\langle Rv, Rw \rangle = \rangle v, w \rangle$ for all $v, w \in \mathbb{R}^2$) and $R^2 = -I$ we have

$$\langle R(X), \nu \rangle = \langle R^2(X), R(\nu) \rangle = \langle -X, R(\nu) \rangle = -\langle X, -\tau \rangle = \langle X, \tau \rangle\,,$$

and thus

$$\int_\Omega \operatorname{curl} X = \sum_{i=1}^{K} \int_{\gamma_i} \langle X, \tau \rangle \, ds = \sum_{i=1}^{K} \oint_{\gamma_i} \langle X, \tau \rangle \, ds\,.$$

□

Remark 8.3.19 (Physical interpretation)**:** The obvious physical interpretation is that the circulation of X around $\partial\Omega$ equals the integral of $\operatorname{curl} X$ over Ω.

A Appendix

A.1 Local surjectivity via the contraction mapping principle

We will briefly explain how one can use the contraction mapping principle to obtain local surjectivity in the proof of the Inverse Function Theorem, Theorem 4.0.4. We use the notation as in the proof there, and we will give a different proof of

Step 2: *f is surjective in a neighborhood around a.*

Note that we can replace $x \mapsto f(x)$ by $x \mapsto f(a+x) - f(a)$. So we can assume $a = 0 \in \mathbb{R}^n$ and $f(a) = 0$. Then (4.4) yields for $g(x) = x - f(x)$

$$\|g(x) - g(y)\| \le \frac{1}{2}\|x - y\| \qquad \forall\, x, y \in B(0, \varepsilon)\,. \tag{A.1}$$

Note that $g(0) = 0$ and so $g(B(0,\varepsilon)) \subset B(0, \varepsilon/2)$. For $y_0 \in B(0, \varepsilon/2)$ define

$$g_{y_0}(x) := g(x) + y_0\,.$$

Note that $g_{y_0}(B(0,\varepsilon)) \subset B(0,\varepsilon)$ and thus by continuity $g_{y_0}(\overline{B(0,\varepsilon)}) \subset \overline{B(0,\varepsilon)}$. Furthermore (A.1) yields (using again continuity of g_{y_0} for the extension to $\overline{B(0,\varepsilon)}$)

$$\|g_{y_0}(x) - g_{y_0}(y)\| \le \frac{1}{2}\|x - y\| \qquad \forall\, x, y \in \overline{B(0,\varepsilon)}\,.$$

Thus $g_{y_0} : \overline{B(0,\varepsilon)} \to \overline{B(0,\varepsilon)}$ is a contraction mapping. Since $(\overline{B(0,\varepsilon)}, \|x-y\|)$ is a complete metric space, the contraction mapping principle yields that g_{y_0} has a unique fixed point $x_0 \in \overline{B(0,\varepsilon)}$. But

$$g_{y_0}(x_0) = x_0 \;\Leftrightarrow\; x_0 - f(x_0) + y_0 = x_0 \;\Leftrightarrow\; f(x_0) = y_0\,.$$

This yields local surjectivity of f.

Bibliography

[1] Michael Spivak, *Calculus on manifolds. A modern approach to classical theorems of advanced calculus*, W. A. Benjamin, Inc., New York-Amsterdam, 1965.