MA241 Lecture Notes, Fall 2023

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CHAPTER 1

Introductory Lecture: What is combinatorics?

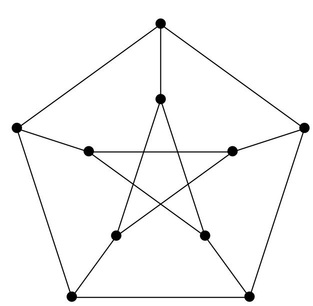
Combinatorics is the study of finite structures. People often think combinatorics means "counting problems," but as we will see (especially later in the module), many of the questions we will ask about finite structures are not about enumerating anything.

Combinatorics is closely related to probability, often by simple linguistics: rather than asking "How many possible 6-digit lottery numbers are there?" one asks "If I choose a random 6-digit number, what is the probability that it is the winning number?"

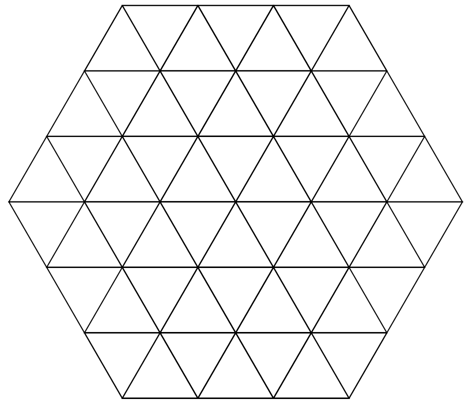
We are about to see many examples of counting problems, some simple and some more complicated. (We will see that a simple-sounding question does not need to have a simple-sounding answer!) Here are two counting problems to get us started:

(1) Suppose I have k identical balls, and n boxes, each box labeled by a number $1, \ldots, n$. How many distinct ways are there to distribute the balls among the boxes (so that every ball goes in a box)?

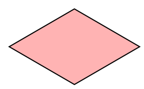
(2) Can you find a path in the following graph that visits each vertex exactly once, and begins and ends at the same vertex? If so, how many such paths?



(3) Suppose I have a regular hexagonal gameboard that looks like this:

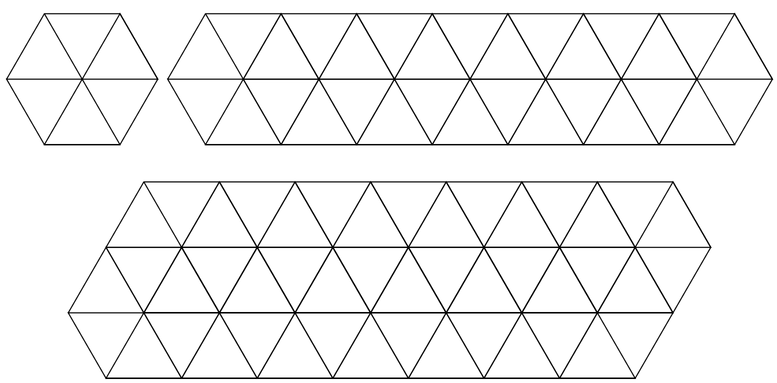


If the side length is n, how many different ways are there to tile the game-board completely with non-overlapping "rhombus domino" pieces as below?



The first problem is a surprisingly useful counting problem; we will solve it, and many variations on the same theme, shortly. The second problem is part of the beautiful and useful field of graph theory, which we will cover in detail later in the term. The third problem is harder, and has deep connections to many other subjects. (E.g. graph theory, linear algebra, complex analysis, algebraic geometry, statistical physics, quantum physics; it is a somewhat simplified model of crystal formation, usually when n is very large.)

Exercise 0.1. How many rhombus domino tilings are there of the following shapes?



(Generalise the last two to arbitrary lengths!)

The answer for the regular-hexagonal board was calculated by Percy MacMahon:

Theorem 0.2 (MacMahon's Formula, 1916). *The number of rhombus domino tilings of the side-length-n hexagon is*

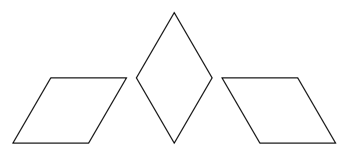
$$\prod_{i=1}^{n}\prod_{j=1}^{n}\prod_{k=1}^{n}\frac{i+j+k-1}{i+j+k-2}.$$

Plugging in $n = 3$ as above yields

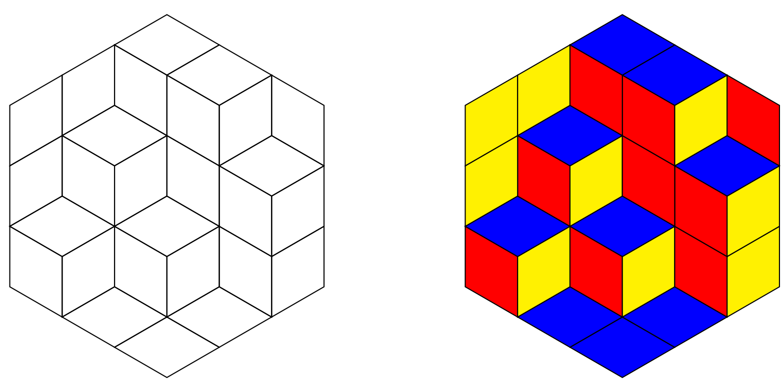
$$\frac{2 \cdot 3^3 \cdot 4^6 \cdot 5^7 \cdot 6^6 \cdot 7^3 \cdot 8}{1 \cdot 2^3 \cdot 3^6 \cdot 4^7 \cdot 5^6 \cdot 6^3 \cdot 7} = 980.$$

While this answer is very beautiful, here are two comments that shows that there is structure to this problem beyond just a counting problem.

Comment 1: There are three orientations that each rhombus can be in:

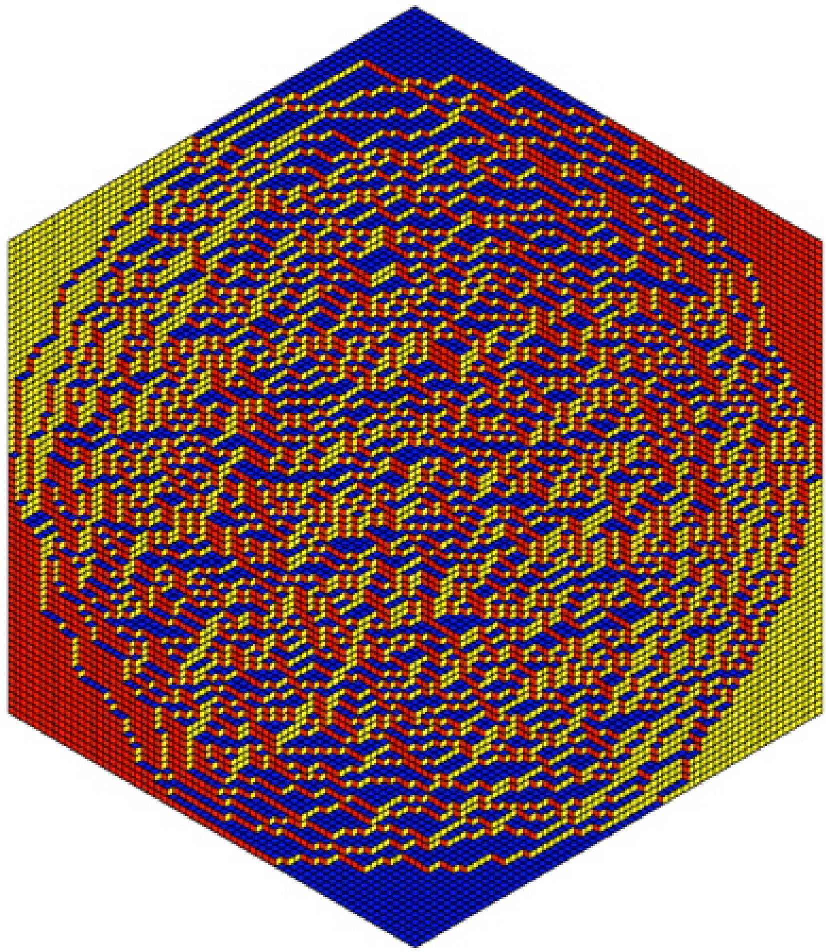


If we tilt our gameboard and colour them differently, we see a 3D effect: box packings in the corner of a room.



In this class, we will often be interested in the relationships between different counting problems; in this case, between our original hexagon-tiling problem and the new problem of counting box packings in the corner of an $n \times n \times n$ room. Often we will solve counting problems using these relationships, where choosing one interpretation over another may give us a different set of proof ideas.

Comment 2: Suppose that of all the tilings of a side-length-n hexagon, we choose a random one. Here is a picture (and this is a typical example):



Note the approximation of a circle around the edge – what causes this? And what might it say about the structural properties of our "crystal"?

CHAPTER 2

Enumerative Combinatorics: Counting Things

1. Placing balls in boxes

We now lay out some of the most basic counting problems, some of which may be familiar.

Example 1.1. Suppose we have k labeled balls (e.g. they have integers $1, \ldots, k$ written on them), and n labeled boxed (labeled with integers $1, \ldots, n$). How many different ways are there to distribute the balls in the boxes. (Every ball has to go in a box, and a box can hold arbitrarily many balls.)

Example 1.2 (Allowing unlabeled balls and/or boxes). Same question, but with k unlabeled (indistinguishable) balls and n unlabeled boxes.

Example 1.3 (Restricting capacities of boxes). Same question, with k unlabeled balls and n labeled boxes, where each box can only hold a single ball. Or each box can hold at most two, or ...

Example 1.4 (Semi-distinguished balls/boxes). Suppose we have k_1 red balls and k_2 blue balls ..., or n_1 green boxes and n_2 yellow boxes ...

Example 1.5 (Most general). We have k_1 balls of colour c_1, k_2 balls of colour c_2, ..., k_r balls of colour c_r. We have n_1 boxes of colour d_1, with capacities $N_{1,1}, \ldots, N_{1,n_1}$, and n_2 boxes of colour d_2, with capacities $N_{2,1}, \ldots, N_{2,n_2}$, ..., n_s boxes of colour d_s, with capacities $N_{s,1}, \ldots, N_{s,n_s}$. How many ways are there to distribute the balls in the boxes?

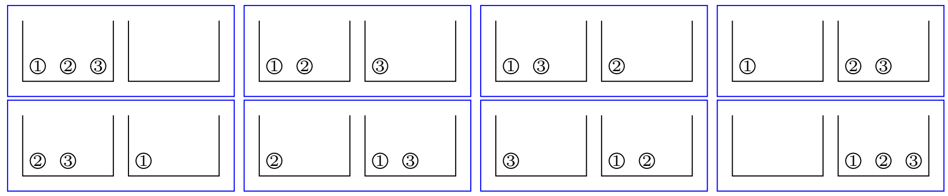
(Even more general: also impose *minimum* capacities for each box, e.g. all boxes must get a ball.)

It is not so important for us to think about the most general cases! However, as we work through special cases, we will see many interesting examples, which we will solve with a variety of different strategies. In doing so, we will build up a toolkit that will equip us to solve a large number of counting problems.

Proposition 1.6. *The number of ways to place k labeled balls into n labeled boxes is n^k.*

Remark 1.7. I have here stated the answer first, and will now prove it – this is backwards from how we usually actually *do* combinatorics, i.e. by using some reasoning to come up with the answer. I'll do both things in this class. In particular, before I prove it, let's see an example.

Example 1.8. Here are the 8 possibilities for $k = 3$, $n = 2$.



Proof of Proposition 1.6. There are n choices for where to put ball 1, n choices for where to put ball 2, etc. □

Remark 1.9. This strategy — considering each ball individually — is probably familiar enough, but is a usually a good way to start any counting problem.

Remark 1.10. If either "labeled" became "unlabeled", problem would be harder! We'll come back to these. (Note to self: unlabeled-labeled is stars-and-bars/weak compositions, labeled-unlabeled is set partitions/Stirling numbers, unlabeled-unlabeled is integer partitions.)

Remark 1.11. How many length-k sequences $(a_1, a_2, \ldots, a_k)$ are there whose entries are $1, \ldots, n$? Same answer, same reasoning. Here they are when $k = 3$ and $n = 2$:

$$\begin{array}{cccc} (1,1,1) & (1,1,2) & (1,2,1) & (1,2,2) \\ (2,1,1) & (2,1,2) & (2,2,1) & (2,2,2). \end{array}$$

If we have two counting problems that seem to be "the same", sometimes we can show a relationship between them by *matching up* the objects being counted. How can you match up the sequences with the ball arrangements? Such a matching up of two sets is called a *bijection*.

2. Counting orderings

Suppose we have k labeled balls and n labeled boxes, and $n = k$, and each box has capacity 1. How many ways can we distribute the balls in the boxes?

Let us apply the previous strategy of considering each ball individually. There are n choices for where to put ball 1. No matter where we put it, there are $n - 1$ remaining choices for where to put ball 2, and so on. Thus the answer is

$$n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1 = \prod_{i=1}^{n} i.$$

This expression will come up so often we give it a name:

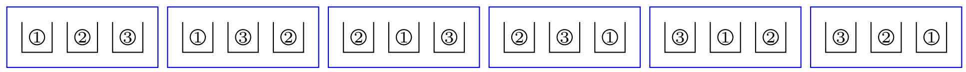
Definition 2.1. Let n be a positive integer. We define the *factorial* of n, written $n!$ and read as "n factorial", to be the product of all positive integers at most n. That is,

$$n! = 1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n = \prod_{i=1}^{n} i.$$

We also define $0! = 1$. (This is a good idea, e.g. because it makes $1! = 1 \cdot (0!)$ true, in accordance with the pattern $n! = n \cdot (n-1)!$.)

(We'll be seeing lots of mathematical definitions – you probably have seen this one before, but it is here as an example of the structure of a mathematical definition.)

Example 2.2. Here are the $6 = 3 \cdot 2 \cdot 1$ choices when $n = 3$.



These distributions are called “reorderings” or “permutations” of $\{1, 2, \ldots, n\}$. Restating the above,

Proposition 2.3. *The number of permutations of* $\{1, 2, \ldots, n\}$ *is* $n!$.

3. Counting subsets two ways

How many ways are there to form a committee of 3 from 10 people? Let’s think through it. Using our previous strategy, let us say there are three slots in the committee. There are 10 people to put in slot 1. Then no matter which person was chosen for slot 1, there are 9 other people for slot 2, then 8 others for slot 3. The answer would seem to be $10 \cdot 9 \cdot 8 = \frac{10!}{7!}$, but we have of course overcounted. In fact, what we have exactly counted is how many committees can be formed when the people are *labeled*, e.g. president/vice president/treasurer.

Luckily, it is easy to tell how badly we have overcounted. In fact, we have counted each committee exactly 6 times, one for each ordering of the members. Thus the correct answer is $\frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = \frac{10!}{7!3!} = 120$.

How can we turn this into a balls/boxes question? There are actually a couple ways to do it. We could have the people be the balls, and have 2 boxes: “On the committee” (capacity 3) and “Not on the committee” (capacity 7). The balls are labeled (as the 10 people are all different people), and the boxes are also labeled.

Alternatively, we could have the people be the boxes, and have the committee slots be the balls. In this case the people/boxes are labeled, whereas the committee slots are not. (Indeed, when we overcounted earlier, the problem was exactly that we had labeled the slots/balls.) Furthermore the boxes now have capacity 1, as a person cannot be chosen for the committee twice. The above counting argument now gives us two new solved problems:

Proposition 3.1. *Let n be a positive integer, and let $k \leq n$ be a nonnegative integer. The number of ways to distribute k unlabeled balls into n labeled capacity-1 boxes is* $\frac{n \cdot (n-1) \cdot \cdots \cdot (n-(k-1))}{k \cdot (k-1) \cdot \cdots \cdot 2 \cdot 1} = \frac{n!}{k!(n-k)!}$.

Proposition 3.2. *Let n be a positive integer, and let $k \leq n$ be a nonnegative integer. The number of ways to distribute n labeled balls into 2 labeled boxes with capacities k and $n - k$ is* $\frac{n!}{k!(n-k)!}$.

Again we give this number a special name:

Definition 3.3. Let $n \geq k$ be positive integers. We define “n choose k”, written $\binom{n}{k}$, to be the quantity $\frac{n!}{k!(n-k)!}$. The number $\binom{n}{k}$ is also called a “binomial coefficient”, for reasons we’ll shortly see.

Remark 3.4. If I had defined this out of the blue, it would not be clear that it is an integer! However it must be an integer by the previous proposition, since it is counting something. (It is surprisingly common in combinatorics to have an expression that you *suspect* is an integer, but in order to prove it, you need to find what counting problem it is the answer to.)

Some notes:

- $\binom{n}{k} = \binom{n}{n-k}$.
- $\binom{n}{0} = \binom{n}{n} = 1$.
- We can write down all the binomial coefficients in "Pascal's triangle":

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$n=0$	1					
$n=1$	1	1				
$n=2$	1	2	1			
$n=3$	1	3	3	1		
$n=4$	1	4	6	4	1	
$n=5$	1	5	10	10	5	1

We might notice the following:

Proposition 3.5. $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Proof. Let us prove this two ways. On one hand, we have

$$\frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{(n-1)!((n-k)+k)}{k!(n-k)!} = \frac{n!}{k!(n-k)!}.$$

On the other hand, suppose we want to choose a k-person committee out of n people. We could either choose Person 1, then choose a $(k-1)$-person committee out of the remaining $(n-1)$ people, or we could *not* chooose Person 1, and make a k-person committee out of the remaining $(n-1)$ people. □

Let us connect the previous counting problem with the one where we counted sequences (or labeled balls, labeled boxes), and at the same time, derive an interesting formula.

Example 3.6. What if the committee were allowed to have any number of people? I.e. distribute 10 labeled balls (people) in 2 labeled boxes with arbitrary capacity: "In the committee" and "Not in the committee". Actually, there is another way to write this as a balls/boxes problem. The committee slots are unlabeled balls, and there are 11 labeled boxes; one for each person (capacity 1), and one labeled "unfilled" (infinite capacity). We have already solved the problem (Proposition 1.6) of labeled balls and labeled boxes with arbitrary capacity: the answer is n^k, i.e. in this case 2^{10}. (Note: One of these choices is the "empty committee".)

On the other hand, we can answer this same question in a different way – it is the number of 0-person committees $1 = \binom{10}{0}$, plus the number of 1-person committees $10 = \binom{10}{1}$, plus the number of 2-person committees $45 = \binom{10}{2}$, and so on. We conclude a lovely formula:

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

For example, $1024 = 2^{10} = 1 + 10 + 45 + 120 + 210 + 252 + 210 + 120 + 45 + 10 + 1$.

Theorem 3.7 (Binomial Theorem). *Let n be a positive integer. Then*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Example 3.8. $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

PROOF. Specifying a term of $(a+b)(a+b)\cdots(a+b)$ consists of choosing, for each factor, either the a or b. There are $\binom{n}{k}$ ways to choose k as and $n-k$ bs.

(Or prove by induction!) □

REMARK 3.9. Plugging in $a = b = 1$ gives the previous formula.

EXAMPLE 3.10. Above, the balls were unlabeled, but we can vary this – perhaps our committee has a 2 co-presidents, and 3 other members. We'll solve this two ways too. We could first pick the co-presidents: there are $\binom{10}{2}$ choices. Then from the remaining 8 people we choose the other members, for $\binom{8}{3}$ choices. The total is

$$\binom{10}{2}\binom{8}{3} = \frac{10!}{2!8!}\frac{8!}{3!5!} = \frac{10!}{2!3!5!}.$$

Alternatively, we could overcount as above (getting $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ choices), and then realize that we are overcounting by a factor of $12 = 2!3!$, corresponding to the orderings of the co-presidents and the other members.

In this problem, the balls came in two "colours" — 2 balls of colour "co-president" and 3 balls of colour "member". (And again, 10 labeled capacity-1 boxes corresponding to people.) We might as well add in a 5 of a third colour of ball "not in committee". We can then summarize:

PROPOSITION 3.11. *Suppose we have n labeled boxes, and n balls — k_1 labeled with colour c_1, k_2 labeled with colour c_2, and so on up to k_r labeled with colour c_r, where $k_1 + k_2 + \cdots + k_r = n$. Then the number of distinct ways of distributing the balls among the boxes is* $\frac{n!}{k_1!k_2!\cdots k_r!}$.

We give this quantity a symbol in analogy with binomial coefficients: it is $\binom{n}{k_1,k_2,\ldots,k_r}$, and it is called a *multinomial coefficient*:

THEOREM 3.12 (Multinomial theorem). *Let n be a positive integer. Then*

$$(a_1 + \cdots + a_r)^n = \sum_{k_1+\cdots+k_r=n} \binom{n}{k_1,\ldots,k_r} a_1^{k_1} \cdots a_r^{k_r}.$$

EXERCISE 3.13. How many ways are there to rearrange the letters in "GRAMMATICAL"? What are the boxes and what are the balls? (HW warm-up)

4. Types of Functions

Recall that if K and N are sets, a *function* $f : K \to N$ assigns a single element of N to each element of K. There are some important classes of functions we'll want to talk about.

DEFINITION 4.1. A function $f : K \to N$ is *injective* or *one-to-one* if it takes each value at most once – that is, if we never have $f(k_1) = f(k_2)$ for different $k_1, k_2 \in K$.

REMARK 4.2. It is only *possible* to have such a function if K has at most as many elements as N, i.e. $|K| \le |N|$.

DEFINITION 4.3. A function $f : K \to N$ is *surjective* or *onto* if it takes each value at least once – that is, if every $n \in N$ is equal to $f(k)$ for some $k \in K$.

REMARK 4.4. It is only *possible* to have such a function if K has at least as many elements as N, i.e. $|K| \ge |N|$.

DEFINITION 4.5. A function that is both injective and surjective is called *bijective*.

REMARK 4.6. It is only *possible* to have such a function if $|K| = |N|$.

REMARK 4.7. A function is bijective if and only if it has an *inverse*; that is, a function $f : K \to N$ is bijective if and only if there exists a function $g : N \to K$ such that $f \circ g = \mathrm{id}_N$ and $g \circ f = \mathrm{id}_K$, where $\mathrm{id}_N : N \to N$ and $\mathrm{id}_K : K \to K$ are the identity functions.

EXAMPLE 4.8. Here is a bijection f from $K = \{1, 2, 3\}$ to $N = \{A, B, C\}$:

$$f(1) = A \quad f(2) = C \quad f(3) = B.$$

4.1. "Bijective proofs". When we want to prove two numbers are equal, we will often find that it is best to find sets of things that the two numbers count, and exhibit a bijection between the sets. (So then they must have the same size, so then the two numbers must be equal.)

For example, consider our two proofs of the fact (Proposition 3.5) that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

One of them was just messing with factorials. The other one said that the left side counts k-element subsets of $\{1, \ldots, n\}$, whereas the other one does too. Why? The first term, $\binom{n-1}{k}$, counts k-element subsets of an $(n-1)$-element set, i.e. $\{2, \ldots, n\}$. The second term, $\binom{n-1}{k-1}$, counts $(k-1)$-element subsets of an $(n-1)$-element set, again $\{2, \ldots, n\}$, which are in bijection with k-element subsets of $\{1, \ldots, n\}$ that contain 1. The second proof is a "bijective proof".

5. Towards inclusion-exclusion – counting surjections

Notation 5.1. For brevity, rather than writing $\{1, \ldots, n\}$ and $\{1, \ldots, k\}$ all the time, we will write $[n]$ and $[k]$. This is reasonably standard. Note that zero is not included, and that $[0]$ is the empty set $\emptyset$.

We have counted functions $[k] \to [n]$ (there are n^k of them) as well as injective functions ($\frac{n!}{(n-k)!}$, assuming $k \leq n$, and zero otherwise) and bijective functions ($n!$, assuming $n = k$, and zero otherwise). What about surjective functions? These are somewhat harder to count! Let's define $\mathrm{Surj}(k, n)$ to be the number.

Example 5.2. $n = 4$, $k = 2$. We could put 3 balls in one box and 1 ball in the other (8 ways). Or we could put 2 balls in each box (6 ways).

(Spends a bunch of time in lecture trying, and failing, to calculate $\mathrm{Surj}(k, n)$ by our standard techniques. We can probably find one or two recursive ways to do it.)

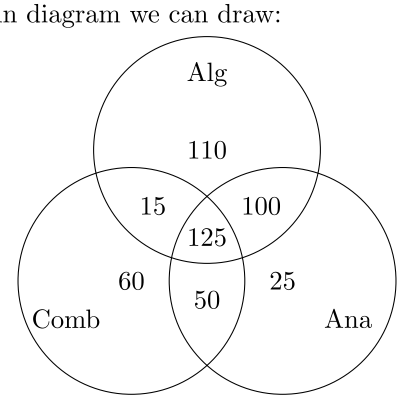
6. Inclusion-Exclusion

Remark 6.1. 250 students are taking Combinatorics and 350 are taking Algebra. How many students are taking at least one of the two? The answer could be anywhere from 350 (if every Combinatorics student is taking Algebra) to 600 (if no student is taking both). To get the right answer, we need to subtract off the number of students taking both, to avoid double-counting them. Summarized: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.

What about $|A_1 \cup A_2 \cup A_3|$? Concretely:

Suppose 250 students are taking Combinatorics, 300 students are taking Analysis, and 350 students are taking Algebra. How many students total are taking at least one of the three? To count, we could add $250 + 300 + 350 = 900$, but we have clearly *overcounted*, because some students got counted more than once.

How badly have we overcounted? Suppose there are 140 students in Combinatorics and Algebra, 225 students in Algebra and Analysis, and 175 students in Analysis and Combinatorics. Furthermore, suppose 125 students are taking all 3 modules. We have counted 125 students 3 times, and $15 + 100 + 50 = 140 + 225 + 175 - 3 \cdot 125 = 165$ students twice. Overall, we have overcounted by $165 + 125 \cdot 2 = 415$, so the total number of students is $900 - 415 = 485$. This consistent with the Venn diagram we can draw:



Regrouping the terms above slightly, we had $900 - (140 + 225 + 175) + 125$. We tried to correct our overcount, but "overcounted" again in our correction!

THEOREM 6.2 (Inclusion-Exclusion Theorem). *Let $A_1, \dots, A_r$ be sets. Then*

$$|A_1 \cup \dots \cup A_r| = \sum_{\substack{I \subseteq \{1,\dots,r\} \\ I \neq \emptyset}} (-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \right|.$$

PROOF. We need to see why each element of $A_1 \cup \dots \cup A_r$ gets counted exactly once in total on the right side. Let $J \subseteq \{1, \dots, r\}$ be the set of A_is containing element x. (In other words, the set of modules student x is enrolled in.) Then x gets counted as +1 in term I for all $I \subseteq J$ with an odd number of elements, and x gets counted as -1 in term I for all nonempty $I \subseteq J$ with an even number of elements. Overall, x gets counted

$$\binom{|J|}{1} - \binom{|J|}{1} + \binom{|J|}{1} - \binom{|J|}{1} + \dots + (-1)^{|J|+1}\binom{|J|}{|J|}$$

times. This number is the same as

$$\binom{|J|}{0} + \sum_{k=0}^{|J|} (-1)^{k+1} \binom{|J|}{k} = \binom{|K|}{0} + 0^{|J|},$$

where the last equality is by expanding $(-1+1)^{|J|}$ using the binomial theorem. Thus x gets counted altogether $\binom{|J|}{0} = 1$ time. □

7. Application of inclusion-exclusion – counting surjections

Back to the problem of counting surjections $[k] \to [n]$. In our earlier failed attempt, we did come across the idea that it might be easier to count functions that are *not* surjections. How can we break up this set as much as possible? How can a function fail to be surjective? It must miss some element $i \in [n]$ — so the set of non-surjective functions is precisely the union, over $i \in [n]$, of functions that miss i. Define A_i to be the set of functions missing i.

Let's try to apply inclusion-exclusion. Can we calculate $|A_i|$? Yes, we can — A_i is just the set of functions $[k] \to [n] \setminus \{i\}$ (with no surjectivity assumption required!), so $|A_i| = (n-1)^k$.

We also need to calculate $|A_i \cap A_j|$. Can we do this? Again, yes — $A_i \cap A_j$ is the set of functions $[k] \to [n] \setminus \{i, j\}$, so $|A_i \cap A_j| = (n-2)^k$.

Similarly we can calculate any term in the inclusion-exclusion formula: $\left|\bigcap_{i \in I} A_i\right| = (n - |I|)^k$. Thus the number of *non*-surjections $[k] \to [n]$ is

$$\sum_{\substack{I \subseteq [n] \\ I \neq \emptyset}} (-1)^{|I|+1} (n - |I|)^k.$$

Subtracting from the total number of functions gives $\mathrm{Surj}(k, n)$:

$$\mathrm{Surj}(k,n) = n^k - \sum_{\substack{I \subseteq [n] \\ I \neq \emptyset}} (-1)^{|I|+1} (n-|I|)^k = \sum_{I \subseteq [n]} (-1)^{|I|} (n - |I|)^k.$$

This is a very nice formula, but perhaps we don't like the fact that it involves a sum over a very large set. We can simplify by grouping terms with the same size I.

Let's do that:

$$\begin{aligned}\mathrm{Surj}(k,n) = \sum_{I\subseteq[n]}(-1)^{|I|}(n-|I|)^k &= \sum_{m=0}^{n}\sum_{\substack{I\subseteq[n]\\|I|=m}}(-1)^{|I|}(n-|I|)^k \\ &= \sum_{m=0}^{n}\sum_{\substack{I\subseteq[n]\\|I|=m}}(-1)^{m}(n-m)^k \\ &= \sum_{m=0}^{n}(-1)^{m}(n-m)^k\sum_{\substack{I\subseteq[n]\\|I|=m}}1 \\ &= \sum_{m=0}^{n}(-1)^{m}(n-m)^k\binom{n}{m}.\end{aligned}$$

Let's check:

$$\mathrm{Surj}(4,2) = 2^4\binom{2}{0} - 1^4\binom{2}{1} + 0^4\binom{2}{2} = 16 - 2 + 0 = 14.$$

Let's also observe that if $k = 1$ and $n > 1$, then

$$0 = \mathrm{Surj}(k,n) = \sum_{m=0}^{n}(-1)^m(n-m)\binom{n}{m}.$$

Using $\binom{n}{m} = \binom{n}{n-m}$ and the substitution $j = n - m$ gives

$$0 = (-1)^n\sum_{j=0}^{n}(-1)^j j\binom{n}{j}$$

— this gives a nice alternate proof to your homework question.

8. Counting set partitions: Stirling numbers

What if we count surjections, but with unlabelled boxes? This has a nice interpretation, namely counting the ways of dividing $[k]$ into n nonempty pieces. These numbers $S(k,n)$ are called the *Stirling numbers of the second kind*. On the other hand, if we think through it carefully, every surjection gets counted exactly $n!$ times. So we have the formula:

$$S(k,n) = \frac{1}{n!}\sum_{m=0}^{n}(-1)^m(n-m)^k\binom{n}{m}.$$

Remark 8.1. Another example where it is not obvious they are integers, or for that matter nonnegative....

Let's write out the Stirling numbers in a triangle:

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$k=1$	1					
$k=2$	1	1				
$k=3$	1	3	1			
$k=4$	1	7	6	1		
$k=5$	1	15	25	10	1	
$k=6$	1	31	90	65	15	1

As with binomial coefficients, there are many observations one can make:

PROPOSITION 8.2. $S(k,n) = S(k-1,n-1) + nS(k-1,n)$.

PROOF. Does $1 \in [k]$ form a singleton block? If yes, there are $S(k-1,n-1)$ ways to partition the rest of the set. If no, removing 1 yields a partitions of $[k] \setminus \{1\} = \{2,\dots,k\}$ into n pieces, and 1 must have come from one of these pieces. □

REMARK 8.3. If we allowed empty boxes, we would be partitioning $[k]$ into *at most* n parts. I do not immediately know of a simpler formula than $\sum_{i=1}^{n} S(k,i)$ for this quantity though...

NOTATION 8.4. For $k \geq 0$, define the Bell numbers $B_k = \sum_{n=0}^{k} S(k,n)$. That is, B_k is the number of ways to partition $[k]$ into any number of parts.

EXAMPLE 8.5. The first few Bell numbers are 1,1,2,5,15,52,203,877,4140,...

PROPOSITION 8.6. $B_k = \sum_{n=0}^{k-1} \binom{k-1}{n} B_n$.

PROOF. Assignment 2. □

9. Not examinable: Application of inclusion-exclusion – counting derangements

This section was not covered in lecture — I decided to only have one application to inclusion-exclusion, namely counting surjections. We know that there are $n!$ permutations of $\{1,\dots,n\}$, i.e. ways of distributing n labeled balls in n labeled boxes of capacity 1. Let us count permutations with the property that ball i is not allowed to be in box i, for any i. These are called derangements.

EXAMPLE 9.1. For $n=3$, the derangements are 2|3|1 and 3|1|2. For $n=4$, the derangements are 2|1|4|3, 2|3|4|1, 2|4|1|3, 3|1|4|2, 3|4|1|2, 3|4|2|1, 4|1|2|3, 4|3|1|2, 4|3|2|1.

Actually, we will count permutations that are *not* derangements — that is permutations for which ball i goes in box i for some i. Let A_1 denote the set of permutations for which ball 1 goes in box 1, A_2 denote the set of permutations for which ball 2 goes in box 2, and so on. We want to calculate $|A_1 \cup \dots \cup A_n|$.

By inclusion/exclusion, we should figure out the sizes of all possible intersections. Indeed, we can see $|A_i| = (n-1)!$, since we need to choose how the other $n-1$ balls are arranged. Similarly $|A_i \cap A_j| = (n-2)!$, and so on. We have

$$|A_1 \cup \dots \cup A_n| = \sum_{\substack{I \subseteq \{1,\dots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1}(n-|I|)!$$

We can simplify this sum by grouping subsets I of the same size:

$$|A_1 \cup \cdots \cup A_n| = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} (n-j)!$$

Thus the number of derangements is

$$n! - \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} (n-j)! = \sum_{j=0}^{n} (-1)^{j} \binom{n}{j} (n-j)! = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!}$$

REMARK 9.2. If n is large, the fraction of permutations that are derangements is about $1/e$. (The probability that if each person at a party takes a random coat on their way out, nobody gets their own coat.)

10. Stars and bars

I briefly mentioned this problem in the first lecture. We have k unlabeled balls, and we have n labeled boxes (of arbitrary capacity). In how many ways can we distribute the balls?

REMARK 10.1. Equivalent formulations: distributing k identical cookies among n people, counting degree-k homogeneous monomials in n variables.

EXAMPLE 10.2. $k = 3$, $n = 3$:

3|0|0	2|1|0	2|0|1	1|2|0	1|1|1
1|0|2	0|3|0	0|2|1	0|1|2	0|0|3

x^3	x^2y	x^2z	xy^2	xyz
xz^2	y^3	y^2z	yz^2	z^3

REMARK 10.3. Note that we get a recursion $C_{k,n} = \sum_{i=0}^{k} C_{k-i,n-1}$.

Strategy: Redraw example.

EXAMPLE 10.4. $k = 3$, $n = 3$:

$*\,*\,*\,\vert\,\vert$	$*\,*\vert\,*\,\vert$	$*\,*\vert\vert*$	$*\,\vert\,*\,*\vert$	$*\,\vert\,*\,\vert*$
$*\,\vert\vert\,*\,*$	$\vert\,*\,*\,*\,\vert$	$\vert\,*\,*\vert*$	$\vert\,*\,\vert\,*\,*$	$\vert\vert\,*\,*\,*$

These are sequences of 5 symbols with two bars and 3 stars! That is, choose 3 stars (and 2 bars) out of 5 slots.

EXERCISE 10.5. Write down a precise bijection between stars/bars sequences and ball distributions.

Conclusion:

PROPOSITION 10.6. *There are* $\binom{n+(k-1)}{n-1}$ *ways to distribute the balls.*

REMARK 10.7. The recursion above is the hockey stick identity!

Remark 10.8. Suppose we want to make sure that each person gets a cookie. (Surjections...) Then we simply start by giving each person a cookie, then carry on as before with the remaining $k-n$ cookies: $\binom{n+(k-n-1)}{n-1} = \binom{k-1}{n-1}$.

Exercise 10.9. Counting problem: k unlabelled balls in n labelled, each box can only hold 1 ball, but I want no two adjacent boxes to have balls in them.

11. Generating Functions

We're now introducing some sequences (and triangles) of special numbers. So far, we've had explicit formulas for all of these (though this won't be the case for the next example...) In practice, we'll often stumble across a new sequence that we don't understand yet. It would be nice to have a systematic way of studying these. First, a note:

Remark 11.1. If you come across a sequence of integers and you want to know more about it, there is an amazing resource called the Online Encyclopedia of Integer Sequences (OEIS). (Don't use it for the assignments unless told to; it will defeat the purpose of some problems.)

However, the point of this section is a more mathematical way of studying infinite sequences of numbers. The difficulty is that, being infinite, they are somewhat difficult to carry around with you. For example, suppose you have two sequences (maybe defined via different counting problems), and you suspect they are equal, but can't find a bijective proof. What do you do? You can compute a bunch of terms and check they agree, but of course you can't prove that two *infinite* sequences are equal in this way.

It is therefore a good idea to find some more finite-looking way of encapsulating the sequence. We will now introduce such a thing, called a generating function. Let me naively rewrite my sequence $1, 1, 2, 5, 15, 52, 203, \ldots$ of Bell numbers as if I were in a calculus class:

$$\frac{1}{0!} + \frac{1x}{1!} + \frac{2x^2}{2!} + \frac{5x^3}{3!} + \frac{15x^4}{4!} + \frac{203x^5}{5!} + \cdots .$$

This is a **formal** power series (where "formal" means we haven't bothered to check whether it has nonzero radius of convergence). You may ask why I've done this, since it seems like we have not gained anything. You may, however, be intrigued to know that the power series above is in fact the Taylor series (based at $x = 0$) for the function e^{e^x-1}. This should be (a) hugely surprising, and (b) a hint at a very powerful tool – this simple function is a much easier object to remember than the infinite sequence!

In fact generating functions are the beginning of a fascinating interplay between analysis and combinatorics. Let's formalize briefly.

Definition 11.2. Let $(a_n)_{n\geq 0}$ be a sequence of numbers. The formal power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is called the *ordinary generating function* of $(a_n)_{n\geq 0}$, and the formal power series

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

is called the *exponential generating function* of $(a_n)_{n\geq 0}$.

REMARK 11.3. The latter is called the exponential generating function because the constant sequence $a_n = 1$ yields the function e^x. Recall from calculus that the ordinary generating function of this sequence is $\frac{1}{1-x}$, as you can see by inspecting the coefficients of $(1-x)(1+x+x^2+x^3+\cdots)$.

I now need to convince you that generating functions are useful. I will do this through a series of examples, beginning with this one.

EXAMPLE 11.4. Let f_n denote the nth Fibonacci number, defined by $f_0 = f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. The first few numbers are (starting with the 0th term): $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots$.

Let's see if we can discover a formula for the generating function $F(x)$ of the Fibonacci numbers:

$$F(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + 21x^7 + \cdots .$$

Note what happens when I multiply $F(x)$ by x or x^2:

$$\begin{aligned} xF(x) &= \quad x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + \cdots \\ x^2F(x) &= \qquad\quad x^2 + x^3 + 2x^4 + 3x^5 + 5x^6 + 8x^7 + \cdots . \end{aligned}$$

Using the recursion we see that $1 + xF(x) + x^2F(x) = F(x)$. We can also see this in formulas, like this:

$$\begin{aligned} F(x) = 1 + x + \sum_{n\geq 2} f_n x^n &= 1 + x + \sum_{n\geq 2} (f_{n-1} + f_{n-2}) x^n \\ &= 1 + x + \sum_{n\geq 2} f_{n-1} x^n + \sum_{n\geq 2} f_{n-2} x^n \\ &= 1 + x + x\sum_{n\geq 2} f_{n-1} x^{n-1} + x^2 \sum_{n\geq 2} f_{n-2} x^{n-2} \\ &= 1 + x + x\sum_{n\geq 1} f_n x^n + x^2 \sum_{n\geq 0} f_n x^n \\ &= 1 + x + x(F(x) - 1) + x^2F(x) = 1 + xF(x) + x^2F(x). \end{aligned}$$

We can solve for $F(x)$ to get $F(x) = \frac{1}{1-x-x^2}$.

At this point, you could use the partial fractions to write $F(x) = \frac{a}{x-\alpha_1} + \frac{b}{x-\alpha_2}$, where α_1, α_2 are the roots of $1 - x - x^2$. If you do this, then apply the geometric series expansion to both terms, you'll get an explicit formula for f_n. I encourage you to work through this.

There is one more thing I want to note about this example. We can write

$$\frac{1}{1-x-x^2} = \frac{1}{1-(x+x^2)} = 1 + (x+x^2) + (x+x^2)^2 + (x+x^2)^3 + \cdots .$$

We should be able to see the Fibonacci numbers from this. Note (See Assignment 1 Problem 4) that f_n is the number of sequences of 1s and 2s with sum n. For example, $f_5 = 8$ because of the eight sequences:

11111 1112 1121 1211 2111 122 212 221

Let's try to reconcile this counting interpretation with the expression

$$F(x) = 1 + (x + x^2) + (x + x^2)^2 + (x + x^2)^3 + (x + x^2)^4 + \cdots .$$

How could we get a contribution to the coefficient of x^5 in $F(x)$? We could get it by distributing out $(x + x^2)^3 = (x + x^2)(x + x^2)(x + x^2)$, by picking an x^2 from two factors and an x from the other. This resembles the sequences $122, 212, 221$ above. We could also get an x^5 by distributing out $(x+x^2)^4$, picking an x^2 from one factor, and an x from the three other factors. The number of ways of doing so correspond to the sequences $1112, 1121, 1211, 2111$. Finally, we could get an x^5 from expanding $(x + x^2)^5$, picking the x from all 5 factors — corresponding to the sequence 11111. So we have a more conceptual explanation for why the counting problem f_n should yield the generating function $\frac{1}{1-(x+x^2)}$.

Over the next few lectures, we'll try to become comfortable with this sort of reasoning — i.e. understanding how the structure of a generating function can correspond directly to a counting problem.

Example 11.5. **Additional example — not covered in lecture:** A population of 50 frogs is introduced into a lake. The population grows by a factor of 4 every year. At the end of each year, 100 frogs are caught and removed. How many frogs are in the lake after n years?

Let a_n denote the answer. We are given the recursion $a_{n+1} = 4a_n - 100$, with $a_0 = 50$. From this you can easily calculate the first few values $50, 100, 300, 1100, 4300, \ldots$. Perhaps with some work you can guess a formula and prove it by induction. (That will certainly not be the case in some future examples.)

Let us see how to solve this in a *systematic* way (i.e. no guessing required) using ordinary generating functions. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Note that

$$\begin{aligned}\sum_{n=0}^{\infty} a_{n+1}x^{n+1} &= \sum_{n=0}^{\infty}(4a_n - 100)x^{n+1} \\ &= 4x\left(\sum_{n=0}^{\infty} a_n x^n\right) - 100x\left(\sum_{n=0}^{\infty} x^n\right) \\ &= 4xA(x) - \frac{100x}{1-x}.\end{aligned}$$

(Note: The identity $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ is familiar from calculus; you can argue it via inspecting coefficients in $(1-x)(1 + x + x^2 + x^3 + \cdots)$.) What about the left side $\sum_{n=0}^{\infty} a_{n+1}x^{n+1}$? We have

$$\sum_{n=0}^{\infty} a_{n+1}x^{n+1} = A(x) - a_0.$$

We thus have

$$A(x) - a_0 = 4xA(x) - \frac{100x}{1-x}.$$

Solving for $A(x)$,

$$A(x) = \frac{a_0}{1-4x} - \frac{100x}{(1-x)(1-4x)}.$$

In some sense we have made progress; we have an explicit formula for $A(x)$. However, you might be unsatisfied with this, because we haven't really got an answer to our question; we need to find the coefficient of x^n in the expression on the right. Let us do so. The first term is

$$\frac{a_0}{1-4x} = a_0 \sum_{n=0}^{\infty} (4x)^n = \sum_{n=0}^{\infty} (50 \cdot 4^n) x^n.$$

The second term requires slightly more work; I'll deal with it via partial fractions, which you may recall from calculus. We attempt to find constants a and b such that

$$\frac{a}{1-x} + \frac{b}{1-4x} = \frac{-100x}{(1-x)(1-4x)}$$

for all x. We need $a(1-4x) + b(1-x) = (a+b) + (-4a-b)x = -100x$. Thus we need $a = -b$, giving $a = 100/3$ and $b = -100/3$. The second term in $A(x)$ thus becomes

$$\begin{aligned} \frac{100}{3(1-x)} - \frac{100}{3(1-4x)} &= \frac{100}{3}\left(\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} 4^n x^n\right) \\ &= \frac{100}{3} \sum_{n=0}^{\infty} (1-4^n) x^n. \end{aligned}$$

Putting it all together,

$$A(x) = \sum_{n=0}^{\infty} \left(50 \cdot 4^n + \frac{100}{3}(1-4^n)\right) x^n.$$

We must have $a_n = 50 \cdot 4^n + \frac{100}{3}(1-4^n) = \frac{50}{3} \cdot 4^n + \frac{100}{3}$. (Exercise: Check that this satisfies the initial condition and the recursion.)

REMARK 11.6. When should I use the ordinary generating function vs. the exponential generating function? The real answer is that either one, or both, or neither, could give you the information you are looking for. (Usually what you're looking for is a relatively simple formula.) Generally, if the sequence grows very fast, you are more likely to have a simple formula if you use the exponential generating function.

12. Some generating functions related to previously seen counting problems

EXAMPLE 12.1. The binomial coefficients are the most basic combinatorial objects in this module. We would like to write down a generating function for them, but we have an immediate problem — they are not really a sequence, they are a triangle. Let's fix k and work with the infinite sequence $\binom{k}{k}, \binom{k+1}{k}, \cdots$.

We want to give a simpler expression for

$$A(x) = \sum_{n=0}^{\infty} \binom{k+n}{k} x^n.$$

I'd like to do it in two different ways. First, we have

$$A(x) = \sum_{n=0}^{\infty} \frac{(k+n)!}{k!n!} x^n.$$

Thus

$$\begin{aligned}
(1-x)A'(x) &= (1-x)\sum_{n=1}^{\infty} \frac{(k+n)!}{k!(n-1)!} x^{n-1} \\
&= \sum_{n=1}^{\infty} \frac{(k+n)!}{k!(n-1)!} x^{n-1} - \sum_{n=1}^{\infty} \frac{(k+n)!}{k!(n-1)!} x^n \\
&= \sum_{n=1}^{\infty} (k+n)\binom{k+n-1}{k} x^{n-1} - \sum_{n=1}^{\infty} n\binom{k+n}{k} x^n \\
&= \sum_{n=0}^{\infty} (k+n+1)\binom{k+n}{k} x^n - \sum_{n=0}^{\infty} n\binom{k+n}{k} x^n \\
&= (k+1)A(x) + \sum_{n=0}^{\infty} n\binom{k+n}{k} x^n - \sum_{n=0}^{\infty} n\binom{k+n}{k} x^n \\
&= (k+1)A(x).
\end{aligned}$$

Solving $A'(x) = \frac{(k+1)}{1-x}A(x)$ has solutions of the form $A(x) = \frac{c}{(1-x)^{k+1}}$, and knowing $A(0) = \binom{k}{k} = 1$ gives $c = 1$. (To convince yourself that I solved the differential correctly, rearrange it to $\frac{A'(x)}{A(x)} = \frac{(k+1)}{1-x}$, and integrate both sides to get $\ln(A(x)) = (k+1)\ln(1-x) + C$.)

Second, and better, let's use combinatorics. Recall ("stars and bars") that $\binom{k+n}{k}$ is the number of ways to divide n cookies among $k+1$ people, or alternatively the number of degree-n monomials in $k+1$ variables. Consider the product

$$(1 + x_1 + x_1^2 + \cdots)(1 + x_2 + x_2^2 + \cdots) \cdot \cdots \cdot (1 + x_{k+1} + x_{k+1}^2 + \cdots).$$

In the expansion of this product, *every* monomial in $k+1$ variables appears exactly once. Setting all variables $x_1, \ldots, x_{k+1}$ equal to each other thus gives $A(x)$. Thus $A(x) = (1 + x + x^2 + \cdots)^{k+1} = \frac{1}{(1-x)^{k+1}}$.

REMARK 12.2. We can also form a generating function in *two* variables:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \binom{k+n}{k} x^n y^k = \frac{1}{1-x-y}.$$

Expanding in k gives

$$\frac{1}{(1-x)-y} = \frac{1}{1-x}\frac{1}{1-\frac{y}{1-x}} = \frac{1}{1-x} + \frac{y}{(1-x)^2} + \frac{y^2}{(1-x)^3} + \cdots$$

as expected.

EXAMPLE 12.3. I claim that the Bell numbers (number of ways to partition the set $[n]$) have exponential generating function e^{e^x-1}. Let's see why.

On Assignment 2, you'll prove $B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k$. Let $A(x) = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$. We have

$$\begin{aligned} A'(x) &= \sum_{n=0}^{\infty} B_n \frac{x^{n-1}}{(n-1)!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \binom{n-1}{k} B_k \frac{x^{n-1}}{(n-1)!} \\ &= \sum_{k=0}^{\infty} \sum_{n=k+1}^{\infty} \binom{n-1}{k} B_k \frac{x^{n-1}}{(n-1)!} \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \binom{k+n}{k} B_k \frac{x^{k+n}}{(k+n)!} \\ &= \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= A(x)e^x. \end{aligned}$$

Thus $\frac{A'(x)}{A(x)} = e^x$. Integrating both sides gives $\ln(A(x)) = e^x + C$, where $C = -1$ is ensured by $A(0) = B_0 = 1$. Thus

$$A(x) = e^{e^x - 1}.$$

13. Integer partitions

Let $k \leq n$ be positive integers. How many ways are there to divide n unlabeled balls into k unlabeled boxes so that each box gets a ball. Now all that matters is the numbers of balls each in box. This is the number of ways to write the integer n as a sum of k positive integers (without caring about the order). We call this number $p_k(n)$.

Example 13.1. There are 8 partitions of 10 into 3 parts:

$$8+1+1 \quad 7+2+1 \quad 6+3+1 \quad 6+2+2 \quad 5+4+1 \quad 5+3+2 \quad 4+4+2 \quad 4+3+3$$

Remark 13.2. Don't get "set partitions" (of $[n]$) and "(integer) partitions" (of n) confused!

Remark 13.3. If we allow any number of boxes, we get $p(n)$, the number of partitions of n

Example 13.4. There are 7 partitions of 5:

$$5 \quad 4+1 \quad 3+2 \quad 3+1+1 \quad 2+2+1 \quad 2+1+1+1 \quad 1+1+1+1+1$$

There is no good formula for $p(n)$ or $p_k(n)$! But let us explore them a little bit more. Here is the triangle for $p_k(n)$.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
$n=1$	1									
$n=2$	1	1								
$n=3$	1	1	1							
$n=4$	1	2	1	1						
$n=5$	1	2	2	1	1					
$n=6$	1	3	3	2	1	1				
$n=7$	1	3	4	3	2	1	1			
$n=8$	1	4	5	5	3	2	1	1		
$n=9$	1	4	7	6	5	3	2	1	1	
$n=10$	1	5	8	9	7	5	3	2	1	1

Here are the first several values of $p(n)$:

$$1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627$$

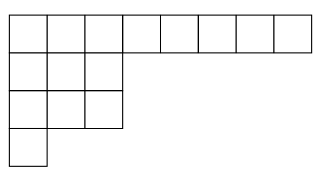
There are many observations we can make about the above:

Exercise 13.5. Show that along and down/right diagonal, the numbers are eventually constant.

Exercise 13.6. Show that $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$. (We observed this in lecture by taking each element, and subtracting the element above and to the left.)

Remark 13.7. We counted "ordered partitions" (sometimes called *compositions*) in the stars and bars section. This is the number of ways to distribute n cookies among k people so that each person gets one, which was $\binom{n-1}{k-1}$. Note the unfortunate reversal of notation — in that section we distributed k cookies among n people...

There is a useful way of drawing partitions. Here is the picture of the partition $15 = 8 + 3 + 3 + 1$:



This is called a *Ferrers diagram* or *Young diagram*.

Remark 13.8. Many of the deep questions in combinatorics can be stated as questions about these diagrams.

The diagrams reveal a useful hidden symmetry among partitions:

Definition 13.9. Let λ be a partition of n. The *conjugate partition* λ^T of λ is the partition corresponding to the reflection of the Ferrers diagram of λ over the (upper-left-to-lower-right) diagonal.

Example 13.10. $4 + 1$ is conjugate to $2 + 1 + 1 + 1$. $3 + 1 + 1$ is conjugate to itself.

Here is an example of how you can use this symmetry, but there are many:

PROPOSITION 13.11. *The number of partitions of n into at most k parts is equal to the number of partitions of n into parts of size at k.*

EXAMPLE 13.12. The partitions of 7 into at most 3 parts:

$$7 \quad 6+1 \quad 5+2 \quad 4+3 \quad 5+1+1 \quad 4+2+1 \quad 3+3+1 \quad 3+2+2$$

The partitions of 7 into parts of size at most 3:

$$3+3+1 \quad 3+2+2 \quad 3+2+1+1$$
$$3+1+1+1+1 \quad 2+2+2+1 \quad 2+2+1+1+1$$
$$2+1+1+1+1+1 \quad 1+1+1+1+1+1+1$$

PROOF OF PROPOSITION 13.11. The two sets of partitions are related by conjugation. □

Here is a slightly trickier bijective argument:

PROPOSITION 13.13. *The number of partitions of n into distinct odd parts is equal to the number of self-conjugate partitions of n.*

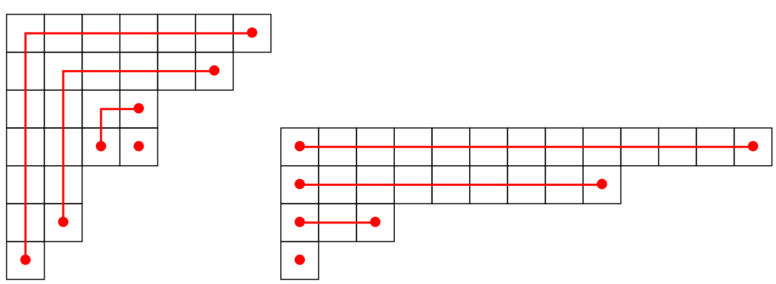
EXERCISE 13.14. Write down the $n = 18$ case. Answer:

$$17+1 \quad 15+3 \quad 13+5 \quad 11+7 \quad 9+5+3+1$$

and

$$9+2+1+1+1+1+1+1+1 \quad 8+3+2+1+1+1+1+1$$
$$7+4+2+2+1+1+1 \quad 6+5+2+2+2+1 \quad 5+4+4+4+1$$

PROOF. Here is a picture – the details are an exercise. The two following pictures get matched:



□

We now start to get into some of the truly tricky combinatorial proofs. This is in some ways the heart of combinatorics. One aspect of this field is that there are simple problems for which extremely difficult, and elegant, arguments are needed. Because the field is so old, many of these have been polished to the point that seem clever to the point of miraculous. This can be quite daunting. As I have emphasized, memorizing these proofs is of limited use. Our goal at the moment is practical: to get you to come up with more and more subtle arguments. Nonetheless, it is worthwhile to present some of the more beautiful arguments as inspiration. I'll do several of these examples over the next few lectures, while still leaving plenty of good problems for the assignments.

14. Deeper properties of partitions.

PROPOSITION 14.1. *The number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.*

EXAMPLE 14.2. $n = 8$. Odd parts:

$7+1 \quad 5+3 \quad 5+1+1+1 \quad 3+3+1+1 \quad 3+1+1+1+1+1 \quad 1+1+1+1+1+1+1+1$

Distinct parts:

$8 \quad 7+1 \quad 6+2 \quad 5+3 \quad 5+2+1 \quad 4+3+1$

I'll give two proofs of Proposition 14.1. In order to do the first one, we need to delve a little deeper into generating functions of partitions.

EXAMPLE 14.3. What is the coefficient of x^n in

$$(1+x+x^2+\cdots)(1+x^2+x^4+\cdots)\cdots(1+x^k+x^{2k}+\cdots)?$$

The answer is the number of ways to write n as a sum of the form $a_1 + 2a_2 + 3a_3 + \cdots + ka_k$. Such expressions are in bijection with partitions of n into parts that are at most k — the bijection is that a_i is the number of copies of i in the partition. Restating what we've just observed:

$$\sum_{n=0}^{\infty} p_{\leq k}(n)x^n = \prod_{i=1}^{k} \frac{1}{1-x^i}.$$

As we are now interested in finding generating functions, this is now a rather stunning observation. (Though I've mentioned that despite the nice expressions for the generating function, there is no closed formula!) If we simply remove the limit on k, we get the equally elegant

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i}.$$

REMARK 14.4. Fun fact (Hardy-Ramanujan):

$$p(n) \sim \frac{1}{4\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

The proof is by analyzing the behavior of the generating function, which actually converges to a well-behaved function. (One actually proves this using complex analysis.)

Let us try to apply the same reasoning to see some more generating functions. What is the coefficient of x^n in

$$(1+x)(1+x^2)(1+x^3)\cdots?$$

It is the number of ways to write n as a sum of *distinct* parts. And what is

$$(1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)\cdots.$$

It is the number of partitions of n into odd parts. But wait – I stated that the number of partitions of n into distinct parts was supposed to be equal to the number of partitions of n into odd parts. Here is the proof.

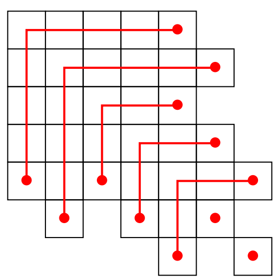
First proof of Proposition 14.1. Using $(1+x^i) = \frac{1-x^{2i}}{1-x^i}$, we have

$$\sum_{n\geq 0} p_{distinct}(n)x^n = \prod_{i\geq 1}(1+x^i) = \prod_{i\geq 1}\frac{1-x^{2i}}{1-x^i}$$
$$= \frac{(1-x^2)(1-x^4)(1-x^6)\cdots}{(1-x)(1-x^2)(1-x^3)\cdots}$$
$$= \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots} = \sum_{n\geq 0} p_{odd}(n)x^n.$$

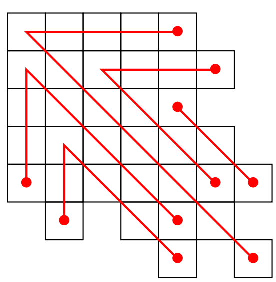
□

I did say I had a second proof. This one is bijective. Again, it is something you might come up with after a *long* bit of thought.

Second proof of Proposition 14.1. Again, I will draw the bijection. Suppose we have a partition of n into odd parts. We can draw its Young diagram, but instead of doing so, we "fold" each row as we did in the last proof, then stack corner-to-corner. For example, the partition $9+9+5+5+5+1+1$ of 35 turns into the following picture:



We now redraw the lines as follows:



The output is $11+8+7+6+3$. It is left to check that the parts of the output must be distinct. (Given a partition into distinct parts, how would you build the last picture? What goes wrong if your partition does not have distinct parts?) □

In fact, there is an unrelated classic bijective proof that is "easier" to come up with, which you can look up if interested.

Finally, I want to mention how one can obtain a beautiful recursion among partitions. Consider the product

$$\prod_{i\geq 0}(1-x^i).$$

If the factors were $1 + x^i$, the coefficients would count the number of partitions of n into distinct parts. Instead, it does that, but weights the partitions into an *odd* number of distinct parts with a -1. That is, the coefficient f_n of x^n is the number of partitions of n into an even number of distinct parts, minus the number of partitions of n into an odd number of distinct parts.

One may show the following:

$$f_n = \begin{cases} (-1)^k & n \text{ is of the form } k(3k \pm 1)/2 \\ 0 & \text{else} \end{cases}.$$

Let's assume we've proved this, and come back to see how it is done in a minute.

Knowing this, we have shown

$$\left(\sum_{n\geq 0} p(n)x^n\right)\left(\sum_{m\geq 0} f_m x^m\right) = 1.$$

We can get a recursion this way. The coefficient of x^n on the left side must be zero, but we can write this coefficient as:

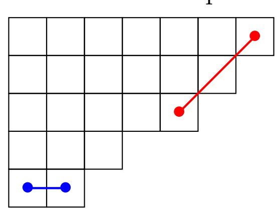
$$\begin{aligned} 0 &= p(n)f_0 + p(n-1)f_1 + p(n-2)f_2 + \cdots + p(1)f_{n-1} + p(0)f_n \\ &= p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - p(n-15) + \cdots \end{aligned}$$

That is,

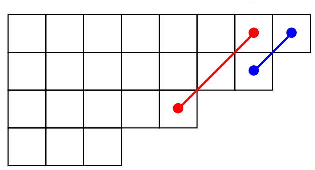
$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \cdots$$

(Note that the numbers $1, 5, 12, 22, \ldots$ are the "pentagonal numbers" $k(3k-1)/2$, and the numbers $2, 7, 15, 26, \ldots$ are the pentagonal numbers shifted by $1, 2, 3, 4, \ldots$, i.e. $k(3k+1)/2$.) This is the pentagonal number theorem, due to Euler. It is the fastest known way to generate the sequence $p(n)$.

Now let's see why f_n is as claimed. This is saying that for "most" n, there is a bijection between partitions of n into an even number of distinct parts and partitions of n into an odd number of distinct parts. Here is the bijection:



Compare the lengths of (a) the bottom row and (b) the longest diagonal passing through the rightmost box. If the row is strictly longer than the diagonal, pull out the diagonal and place it below as a new row. (We still have distinct parts, and one more.) If the diagonal is at least as long as the row, pull out the row and place it to the right of the diagonal. (We still have distinct parts, and one fewer.)



This is reversible (in fact, it is its own inverse), as in case (i), the new longest diagonal will be at least as long as the new row, and in case (ii), the new longest row will be longer than the diagonal. This seems like we are done – but where did we use the hypothesis that n is not of the form $k(3k \pm 1)/2$?

There are two partitions to which this does not apply. As an example with 3 rows, they are: $5+4+3$ and $6+5+4$. In the first case, both lengths are 3, but we cannot move the bottom row up. In the second case, the diagonal has length 3, but if we move it down, we get $5+4+3+3$, which does not have distinct parts.

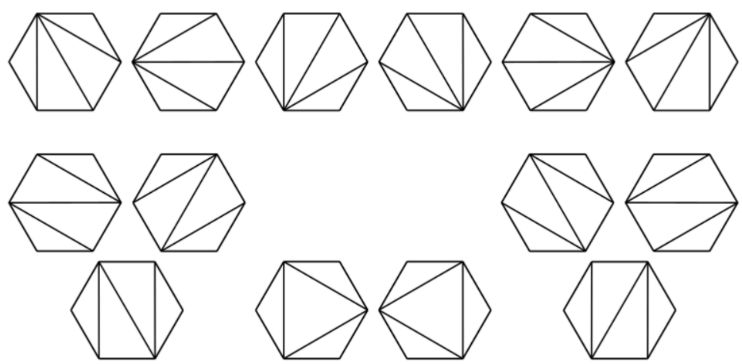
Each of these give an *extra* partition with $k = 3$ rows that is not matched up via the bijection – partitions of $5+4+3 = 12 = k(3k-1)/2$ and $6+5+4 = 15 = k(3k+1)/2$ respectively. Generalizing, we get an extra partition with k rows (contributing $(-1)^k$) of $n = k(3k \pm 1)/2$.

15. Catalan numbers

We will soon finish the part of the module devoted to counting problems; we end with one last one, which you'll explore a bit further on Assignment 3.

DEFINITION 15.1. A *triangulation* of a regular n-gon is a collection of non-crossing diagonals that divide the n-gon into triangles.

Here are the 14 triangulations of a regular hexagon:



DEFINITION 15.2. The nth *Catalan number* C_n is the number of triangulations of a regular $(n+2)$-gon.

The first few Catalan numbers are

n	0	1	2	3	4	5	6	7	8
C(n)	1	1	2	5	14	42	132	429	1430

EXERCISE 15.3. The Catalan numbers satisfy $C_n = \sum_{i=0}^{n-1} C_i C_{n-i}$. (Hint: Label one edge of the $(n+2)$-gon, and keep track of which triangle of the triangulation contains that edge.)

EXERCISE 15.4. Use the above recursion to show that the generating function of the Catalan numbers is

$$A(x) = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x}.$$

(Hint: How would you expand out $A(x)^2$?)

Remark 15.5. In lecture I didn't explain clearly why we get the negative branch of the square root. The reason is that if we chose the positive branch, the limit $\lim_{x\to 0} A(x)$ would not exist. (Note that this isn't 100% rigorous since we haven't been completely careful about how formal power series work. In particular, we can make sense of the expression $\frac{1-\sqrt{1-4x}}{2x}$ as a formal power series, whereas we can't do so for $\frac{1+\sqrt{1-4x}}{2x}$ — though the latter can actually be written as the "Laurent series" $\frac{1}{x} - 1 - x - 2x^2 - 5x^3 - \cdots$.)

Definition 15.6. A *balanced sequence* n opening and n closing parentheses is a sequence such that every close-paren has a matching open-paren.

Here are the 14 balanced sequence of 4 pairs of parentheses:

(((())))	((()()))	((())())	((()))()	(()(()))	(()()())	(()())()
(())(())	(())()()	()((()))	()(()())	()(())()	()()(())	()()()()

Exercise 15.7. Prove that C_n counts the number of sequences of n opening and n closing parentheses, such that every close-paren has a matching open-paren. Do this either by proving that balanced sequence of parentheses satisfy the recursion 15.3, or by finding a bijection between balanced sequences and triangulations.

Exercise 15.8. Prove that C_n counts north-east lattice paths from $(0, 0)$ to (n, n) (as in Assignment 1) *that never cross strictly above the diagonal*. These are called *Dyck paths*.

The Catalan numbers count a surprisingly large number of things. The textbook *Enumerative Combinatorics, Vol.2*, by Richard Stanley, has a famous exercise that gives 66 different counting problems for which the answer is the Catalan numbers. Stanley later wrote a book including a vastly expanded list, with 214 different counting problems for which the answer is the Catalan numbers.

There are now a few ways to obtain a formula for C_n.

(1) There is a nice proof that involves using the recursion above to find the generating function
$$\sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}.$$
One then Taylor expands to find a formula for C_n.

(2) Let's do it bijectively:

Instead of counting balanced sequences of n pairs of parentheses, we count those that are *not* balanced — let's call these "bad sequences". (We know that there are $\binom{2n}{n}$ sequences total.)

Given a bad sequence, it has some first unmatched close-paren. Consider the following operation: for all parens *strictly after* the first unmatched close-paren, we switch each paren, making every open-paren a close-paren and vice versa. Now, we have $n-1$ open-parens and $n+1$ close-parens.

Now, I claim that this operation gives a bijection between bad sequences of n pairs of parentheses and *all* sequences of $n-1$ open-parens and $n+1$ close-parens. That is, I need to show the operation is reversible. I will describe the reverse operation. Given a sequence of $n-1$ open-parens and $n+1$ close-parens, it must have an unmatched close-paren, since it has more close-parens than open-parens. It therefore has a first unmatched close-paren. Do the same switching operation to everything strictly after this first unmatched close-paren. This is clearly the inverse of the above operation, and clearly outputs a bad sequence since the first unmatched close-paren is still unmatched.

This bijection shows that there are $\binom{2n}{n-1}$ bad paths. Thus

$$\begin{aligned} C_n = \binom{2n}{n} - \binom{2n}{n-1} &= \frac{2n!}{n!n!} - \frac{2n!}{(n-1)!(n+1)!} \\ &= \frac{2n!(n+1-n)}{n!(n+1)!} \\ &= \frac{2n!}{n!(n+1)!} = \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$

CHAPTER 3

Graph Theory

1. Motivation

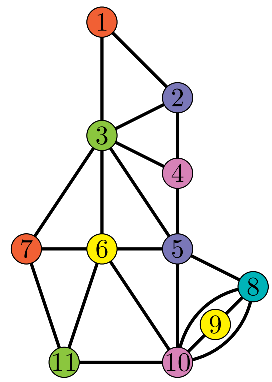
On a map, different countries or regions are often shaded with different colours in order to make the borders easy to see. For example, here is a map I found online of the regions of the UK.

Note that this map has been coloured with 7 colours (including the ocean). Let's focus on the main island (Britain). Does it really need 6 colours, or could we colour it with fewer? We can convince ourselves that it is impossible to colour with 3 colours, and that it is possible to colour it with 4.

At least as early as the 1850's, mathematicians were aware of the question: Can we colour every map with at most four colours? (One has to give a proper definition of what counts as a map – an important caveat is that regions are not allowed to have two separate pieces.) For over 120 years, nobody knew the answer, though many mathematicians (including some very famous ones) gave false proofs.

One conceptual breakthrough required is that the shape of the countries is irrelevant — the *only* information about the map that we need is which countries are *adjacent* each other. *Graphs* are a fundamental mathematical object designed to capture the concept of adjacency. We replace each region of our map with a *vertex*, and if two regions are adjacent, we draw an *edge* between the two vertices.

There are a few subtleties; for example, *adjacent* has to mean that they cannot just touch at a single point. This guarantees that I can draw the graph without having any of the edges cross each other. Also I've drawn two edges between Anglia and Southeast England since they touch twice.



Now I have turned my problem into combinatorics, since the graph contains finite data: A finite set of vertices, and a finite set of edges. (The shape of an edge is not part of the data — only which two vertices it connects.) The question now becomes:

Let G be a graph that can be drawn in the plane with no crossing edges (i.e. G is planar — we'll discuss this much more later). Then is it possible to colour the vertices with 4 colours such that if two vertices are adjacent (connected by an edge), then they get different colours?

The answer is yes! This is the **Four colour Theorem**, proved by Appel and Haken around 1976. This proof was controversial, as it involved extensive case-by-case checking (1834 cases) that could not be done by hand, but had to be verified by computer. Even today, no non-computer-assisted proof is known. (The validity of the computer-assisted proof has been thoroughly checked by many people though.)

Graphs turn out to be a very versatile concept, and they are used to model all sorts of phenomena. The rest of the module will be spent on the theory of graphs — a very broad field containing a huge number of different types of interesting problems. (Relevant future modules include MA252 Combinatorial Optimization and MA4J3 Graph Theory.) We will however have time to introduce many of most ubiquitous graph-theoretic problems/ideas, and to prove a bunch of interesting things.

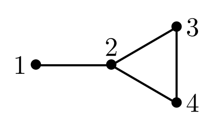
2. The language of graphs

Formally:

DEFINITION 2.1. A *graph* (or *simple graph*) $G = (V, E)$ is

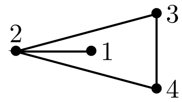
- a set V, whose elements are called the *vertices* of G, and
- a set E of unordered pairs of distinct vertices, whose elements are called the *edges* of G.

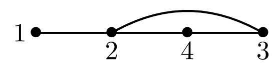
Here is an example:



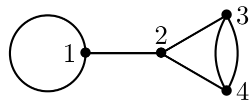
Here $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$. (Vertices should *always* be labeled!) While "infinite graphs" are also important in some applications, most graphs we see will have finite vertex and edge sets; we will assume this is the case unless otherwise specified.

As you see from the definition, a graph only encodes "connectivity", e.g. the previous graph could also be drawn in the following ways, and it is the *same* graph:





For now, we do **not** allow duplicate edges and loops, as in:



Occasionally, however, we will need to do so — we will always be careful to specify, and will refer to this as a *multigraph* to avoid confusion. In this case we would widen the definition of E to be a *multiset*, where the two elements of an edge need not be distinct.

It is confusing to choose a convention that is at odds with half of the literature — so instead, I will just try to be very clear, in every scenario, about what I mean. Luckily, for most of our purposes, we will only need to mention simple graphs – this is because simple graphs really are all about encoding the concept of adjacency, whereas duplicate edges and loops don't have any bearing on the question of whether two vertices are adjacent.

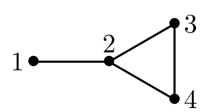
Here is a bit of terminology that you need to now internalize:

DEFINITION 2.2. Let $G = (V, E)$ be a graph.

- If $e \in E$ contains v, we say *e is incident to v*, or *v is an endpoint of e*.
- If $v_1, v_2 \in V$ are vertices such that $\{v_1, v_2\} \in E$, we say v_1 and v_2 are *adjacent* (or are *neighbours*).
- For $v \in V$, the number of edges incident to v is called the *degree* $\deg(v)$ or *valence* $\mathrm{val}(v)$ of v.

DEFINITION 2.3. A graph is called *k-regular* if every vertex has degree k. (Or just *regular* if it is k-regular for some k.)

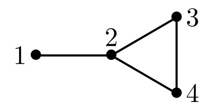
EXAMPLE 2.4. In the graph below, vertex 1 has degree/valence 1, vertex 2 has degree 3, and vertices 3 and 4 have degree 2.



Graphs are a very rich structure, and graph theory forms a huge part of modern mathematical research. Unlike in the first part of the module, where we solved a range of somewhat-disparate counting problems (though we did see many recurring themes), in this part of the module we really will build more of a theory, and try to gain an intuition for how graphs can and cannot behave. We will also see many more examples very soon. To give you a taste of how one might start proving theorems about graphs, let us observe:

PROPOSITION 2.5 (Degree-sum formula). *Let $G = (V, E)$ be a graph. Then $\sum_{v \in V} \deg(v) = 2|E|$. In particular, the number of odd-degree vertices of G is even.*

EXAMPLE 2.6. We had 2 such vertices in the example, namely 1 and 2.

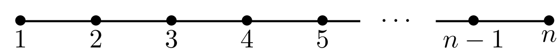


PROOF OF PROPOSITION 2.5. Consider $\sum_{v \in V} \deg(v)$. Each edge contributes exactly 2 to this sum, i.e. $\sum_{v \in V} \deg(v) = \sum_{e \in E} 2 = 2\,|E|$. Thus the left side must be even, from which the statement follows. $\square$

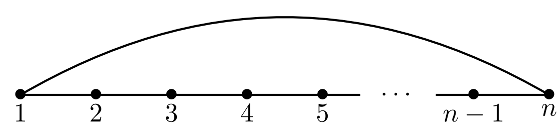
(If you like, we just counted the size of the set $H = \{(v, e) \in V \times E : e \text{ incident to } v\}$ in two different ways. H is sometimes called the incidence set of G.)

Here are a few of the most basic graphs.

- The *complete graph on n vertices*, denoted K_n, is the graph (V, E) where $V = [n]$ and $E = \binom{[n]}{2}$, where $\binom{[n]}{2}$ is (*new* notation for) the set of 2-element subsets of $[n]$. That is, all possible edges are drawn in, so any two vertices are adjacent.
- The *empty graph on n vertices* has n vertices and no edges.
- The *path graph on n vertices*, denoted P_n, is the graph



- The *cycle graph on n vertices*, where $n \geq 1$, denoted C_n, or *n-cycle*, is the graph



(We could draw it as a regular n-gon.)
- This is not an example, but a way of generating examples. The research area of Random Graph Theory is currently quite active. As an example, take n vertices, and for each pair, flip a coin to determine whether or not to draw an edge. (Or better yet, draw an edge with some probability $0 \leq p \leq 1$.) What is the probability that the resulting graph will "contain a K_3"? (I.e. what is the probability you can find 3 vertices that are all connected to each other?) This will be a function of n and p – questions like this often have interesting applications, since lots of interactions can be modeled by random graphs.

  (Can you think of other algorithms to generate random graphs?)

We'll see lots more examples!

3. The language of graphs, continued

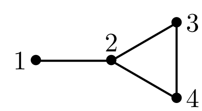
DEFINITION 3.1.
- Two (simple) graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there exists a bijection $\phi : V \to V'$ such that $\{v_1, v_2\} \in E$ if and only if $\{\phi(v_1), \phi(v_2)\} \in E'$. (That is, if they differ only in the naming of the vertices.)
- Let $G = (V, E)$ and $H = (V', E')$ be simple graphs, where $V' \subseteq V$. We say H is a *subgraph* of G if for all edges $\{v_1, v_2\} \in E'$, we have $\{v_1, v_2\} \in E$.

- Let $G = (V, E)$ be a connected graph, and let $H = (V', E')$ be a subgraph (so $V' \subseteq V$). We say H is a *spanning subgraph* (or H *spans* G) if $V' = V$.
- For a graph $G = (V, E)$ and a subset $V' \subseteq V$, the *subgraph of G induced by V'* is the subgraph with vertex set V' and edge set $E' = \{e \in E : e \subseteq V'\}$. That is, the set of all edges between vertices in V'.

REMARK 3.2. The problem of determining whether two graphs G and G' are isomorphic is "algorithmically difficult". There is a brute-force algorithm — for each bijection between their vertex sets, see if the edge sets match — but it is not known whether there is an "efficient" algorithm, i.e. one that returns an answer in an amount of time that is a polynomial function of $|V|$.

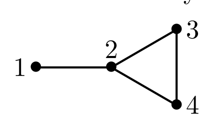
Sometimes, we will care about specific bijections/injections, not just whether one exists. For example, an *automorphism* of a graph G is an isomorphism from G to itself (a permutation of V that preserves the graph structure). Automorphisms of a graph form a group $\mathrm{Aut}(G)$.

EXAMPLE 3.3. In our running example, what are the automorphisms? There is the identity, and there is the bijection $(1, 2, 3, 4) \mapsto (1, 2, 4, 3)$.



DEFINITION 3.4. Let G and G' be graphs. We say G' contains G if G is isomorphic to a subgraph of G'.

EXAMPLE 3.5. The graph below contains two paths *from vertex 1 to vertex 4*. It contains a 3-cycle, but does not contain a 4-cycle.

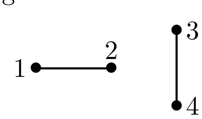


DEFINITION 3.6. A graph $G = (V, E)$ is *connected* if for every pair $v_1, v_2 \in V$, there is a path in G from v_1 to v_2. (Usually a good property if your graph represents, e.g., an airline, or the internet.) The maximal connected subgraphs of a (disconnected) graph are called its *connected components*.

REMARK 3.7. A path in G from v_1 to v_2 is formally defined above, meaning you can walk from v_1 to v_2 along edges of G *without repeating vertices* (or, therefore, edges). We should note however, that this is *not* a stronger condition than being able to walk from v_1 to v_2 *with* possibly repeated vertices. Given such a walk, if you come to a vertex that will be visited multiple times, just skip the intervening steps.

Note: A *walk* is used to mean "path with possibly repeated vertices/edges."

EXAMPLE 3.8. The following is not connected:

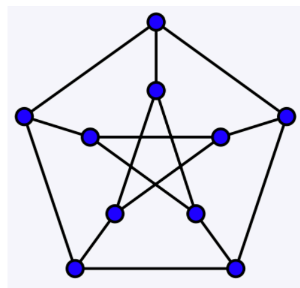


EXERCISE 3.9. Find a connected 4-regular graph with 6 vertices. (Can you find one where the edges don't cross each other?)

4. Basic measurements of graphs

DEFINITION 4.1. An *independent set of vertices* in a graph is a subset of the vertices such that no two elements are adjacent. The *vertex independence number* $\text{ind}_V(G)$ of G is the cardinality of the largest independent set in G.

EXERCISE 4.2. Compute $\text{ind}_V(G)$ for G the *Petersen graph*, below:



DEFINITION 4.3. A *matching* in a graph is a subset of the edges such that no two elements share an endpoint. The *matching number* $\text{ind}_E(G)$ of G is the cardinality of the largest matching in G. If there exists a matching using all vertices of G, it is called a perfect matching.

EXERCISE 4.4. Compute $\text{ind}_E(G)$ for G the Petersen graph.

DEFINITION 4.5. An *colouring* of a graph $G = (V, E)$ with colour set C is a function $f : V \to C$ such that for every edge e of G, the endpoints of e are different colours (i.e. if the endpoints are called v_1 and v_2, then $f(v_1) \neq f(v_2)$). The *chromatic number* $\chi(G)$ of G is the cardinality of the smallest colour set for which there exists a colouring of G.

EXERCISE 4.6. Compute $\chi(G)$ for G the Petersen graph.

DEFINITION 4.7. The distance between two vertices v_1, v_2 in a graph $G = (V, E)$ is the length of the shortest path containing v_1 and v_2. The diameter $\text{diam}(G)$ of G is the maximum distance between two vertices in G.

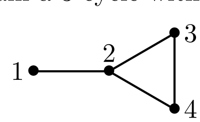
EXERCISE 4.8. Compute $\text{diam}(G)$ for G the Petersen graph.

DEFINITION 4.9. The *girth* $\text{girth}(G)$ of a graph $G = (V, E)$ is the length of the shortest cycle in G.

EXERCISE 4.10. Compute $\text{girth}(G)$ for G the Petersen graph.

5. Trees

DEFINITION 5.1. A graph G is called *acyclic* if it does not contain any cycles. That is, there is no subgraph isomorphic to C_n for any $n \geq 1$. Our running example is not acyclic, since it does contain a 3-cycle with vertices 2,3,4.

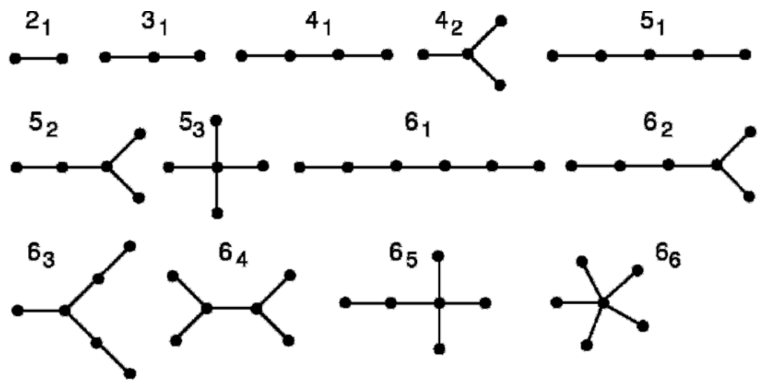


In particular, G cannot contain loops or double edges: acyclic graphs are simple.

Definition 5.2. A connected acyclic graph is called a *tree*. (A disconnected acyclic graph, i.e. a union of trees, is called a *forest*.)

Remark 5.3. Slightly different from the computer science version of a tree, which might include a specified vertex called the *root*. We will refer to that as a rooted tree.

Example 5.4. Here are some trees:



Trees are going to be a huge part of the rest of the module. We will next prove some basic theorems about trees. But first, a counting problem.

Example 5.5. How many trees are there with vertex set $[6]$?

We can see the different types of trees in the figure about, and you can convince yourself that these are all the possibilities. We need to count how many ways there are to label the vertices with $1, \dots, 6$ *that give different trees*. Check yourself: for the six types we get

$$360 + 360 + 360 + 90 + 120 + 6 = 1296.$$

Theorem 5.6. *If $G = (V, E)$ is a tree, and $e = \{v_1, v_2\} \in E$ is an edge, then $G' = (V, E \setminus e)$ is not connected. ("Cutting any edge disconnects the tree.")*

Proof. Suppose G' is connected. Then there exists a path P from v_1 to v_2 in G'. As P is contained in G', it does not include the edge e; adding the edge e yields a cycle contained in G, a contradiction. □

In fact, trees are exactly the "edge-minimal connected graphs":

Theorem 5.7. *Conversely, if G is connected, and cutting any edge disconnects G, then G is a tree.*

Proof. Suppose G is connected, and cutting any edge disconnects G. We must show that G is acyclic. Suppose for contradiction that G contains a cycle C. Cut any edge e of C. By assumption, this disconnects G, yielding a graph G', so there exist v_1, v_2 not connected by a path in G'. In other words, any path from v_1 to v_2 in G must have included e. But this is impossible – take such a path, and replace e with "going around C the long way". This yields a path from v_1 to v_2 that does not contain e. (Technically this could be a walk from v_1 to v_2, as our original path may have already used vertices of C, which would now be repeated, but as observed in Remark 3.7, it can be easily turned into a path by deleting redundant segments.) □

Based on the examples above, let us conjecture:

Conjecture 5.8. *A tree with n vertices has $n-1$ edges.*

In order to prove this, we would like to induct as follows. A *leaf* of a tree is a degree-1 vertex. Given a tree, we would like to remove a leaf and its adjoining edge, then appeal to the inductive hypothesis.

Theorem 5.9. *A (finite) tree with at least two vertices has at least one leaf. (In fact, at least two!)*

Proof. Idea: Sitting in the middle of a tree, how would you find a leaf? You would keep walking until you hit a dead end. Your path cannot cycle, so you must hit a dead end, and that dead end would have to be a leaf.

This is almost a proof, but we want to be a bit more careful about what it means to "keep walking until you hit a dead end".

By assumption G has at least two vertices, so G contains at least one path. Let P be a *maximal* path in G — a path that is not contained in any other path. (Such a path must exist, as otherwise one could keep building paths with successively more and more vertices — but we have only finitely many vertices.) The path P has two endpoints, call them v_1 and v_2. As they are the endpoints of P, we have $\deg(v_1), \deg(v_2) \geq 1$. We claim that v_1 and v_2 are leaves.

Suppose not, e.g. suppose $\deg(v_1) \geq 2$. Then there is some edge $e' \notin P$ incident to v_1. Let v' denote the other endpoint of e'. As P is maximal, we know that we *cannot* extend P by adding e' to get a longer path. That is, we know that v' is already in P. This gives us a cycle: First, traverse P from v' to v_1, then take the edge e'. This contradicts the assumption that G is a tree, hence acyclic. Thus v_1 and v_2 are leaves. □

Remark 5.10. An infinite tree can have 0 or 1 leaves...

Now we can prove:

Theorem 5.11. *A tree with n vertices has $n-1$ edges.*

Proof. Let G be a tree with n vertices. If $n = 1$, an edge would have to be a loop, which are not allowed as G is acyclic. Thus there are no edges, so the base case holds.

If $n > 1$, let v be a leaf of G, with sole incident edge e. Let G' be the graph obtained by removing v and e. Certainly G' is still connected, and still acyclic. By the inductive hypothesis, G' has $n-1$ vertices and $n-2$ edges. Thus G has $n-1$ edges. □

What about the converse? Try to draw a graph with n vertices and $n-1$ edges that is not a tree — it will end up not being connected...

Theorem 5.12. *A connected graph with n vertices and $n-1$ edges is a tree.*

We can prove it using the characterization of trees as minimal connected graphs.

Proof. Let G be a connected graph with n vertices and $n-1$ edges. We want to start deleting edges (but not vertices), if possible without disconnecting the graph, until we cannot anymore. To use the better logic we established in Theorem 5.9, we observe that G has a connected spanning subgraph — namely, G itself. Among all connected spanning subgraphs, we then choose one, call it H, with as few edges as possible — this must be possible since G has finitely many edges. By construction, if we delete an edge of H, it becomes disconnected, so by Theorem 5.7, we know H is a tree. By Theorem 5.11, H has $n-1$ edges. Since H was obtained from G by deleting edges, we conclude that actually $H = G$. Thus G is a tree. □

REMARK 5.13. If G has n vertices and *fewer* than $n-1$ edges, G cannot be connected. If it were, we would go through the exact proof above, but find that H has *more* edges than G, a contradiction.

Alternatively, and intuitively, we can see this by starting with n vertices and no edges, then adding edges one by one. We start with n connected components. Each added edge reduces the number of connected components by at most 1. In particular, if G has n vertices and k edges, then G has *at least* $n-k$ connected components. The minimum number of connected components, i.e. $n-k$, is precisely the case where G is acyclic, i.e. a forest.

To summarize, we have:

THEOREM 5.14. *Let $G = (V, E)$ be a graph. The following are equivalent:*

(1) G is a tree — that is, G is connected and acyclic.
(2) G is minimally connected *— that is, G is connected, but removing any edge disconnects G.*
(3) G is connected and satisfies $|E| = |V| - 1$.

In fact, we could add to this list the statement "Any two vertices in G are connected via a unique path" — see Assignment 3.

Trees will come back regularly for the rest of the term; among other things, we'll see some interesting counting problems involving them. For now, we note a few useful definition/observations:

OBSERVATION 5.15. *Given a tree T, and a vertex v of T, we may give T the structure of a* directed graph *(where we give each edge a direction) by assigning all directions to point "away" from v. Precisely, let e be an edge, connecting vertices v_1 and v_2. There are unique paths from v to both v_1 and v_2, and exactly one of them (say from v to v_2) contains e. Then e is given direction $v_1 \to v_2$. (This is the process of "choosing a root" for T.)*

OBSERVATION 5.16. *Every connected graph G contains a spanning tree T. We already saw the argument — just delete edges, keeping the graph disconnected, until you cannot do so anymore. More precisely, let T be any minimal connected spanning subgraph.*

6. More on colourings of graphs

Recall that a colouring of a graph is an assignment of a colour to each vertex such that no edge connects two vertices of the same colour. A k-colouring is a colouring wth k colours. Recall also that the chromatic number $\chi(G)$ of a graph G is the fewest colours needed for a colouring.

REMARK 6.1. Graph colouring is often used to model scheduling problems. Suppose you have n committees, each of which needs to have a 1-hour meeting. Some of the meetings cannot overlap, since the committees might overlap. Draw a graph with n vertices, where edges correspond to overlapping committees. Then "colour" the graph with time slots. The colouring condition ensures that no two overlapping committees meet in the same time slot. The chromatic number is the smallest amount of hours in which the committees can meet.

There are many many more applications — your committee meetings could e.g. be replaced by tasks to be performed by a parallel computer, some of which require access to the same file, so shouldn't be done simultaneously.

Here is an observation that gives us an easy upper bound on the chromatic number of G.

OBSERVATION 6.2. *If the vertices of G are coloured with k colours, and there exist two colour classes with no edges between them, then we could just merge those colours to get fewer. So if G is coloured with $\chi(G)$ colours (as few as possible), we must have that any two colour classes have an edge between them. This gives $\binom{\chi(G)}{2}$ edges – in particular G must* have *at least this many edges. That is, $|E| \geq \frac{1}{2}\chi(G)(\chi(G)-1)$. Rearranging this proves:*

PROPOSITION 6.3. *If G has m edges, then $\chi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$.*

Here is another one:

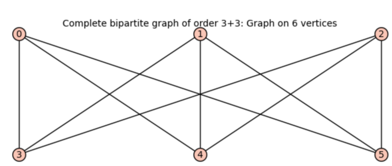
PROPOSITION 6.4. $\chi(G) \leq (\max_{v \in V} deg(v)) + 1$.

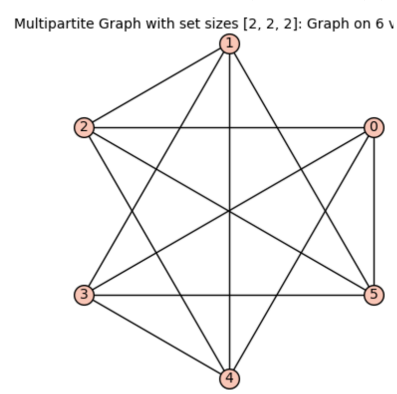
PROOF SKETCH. Colour the vertices in some order – at each step, there is an available colour. □

DEFINITION 6.5. A graph that has a 2-colouring is called *bipartite*. That is, G is bipartite if $\chi(G) \leq 2$. Sometimes G is called k-partite if $\chi(G) \leq k$.

EXAMPLE 6.6. The *complete bipartite graph* $K_{a,b}$ has vertex set $V = V_1 \sqcup V_2$, where $|V_1| = a$, $|V_2| = b$, and there is a single edge connecting every vertex of V_1 to every vertex of V_2. (So ab edges altogether.) These will show up quite a bit.

One can also define the complete multipartite graph $K_{a_1,a_2,\ldots,a_r}$, in which vertices are divided into partite sets of sizes $a_1, \ldots, a_r$, and vertices are connected by an edge if they are in different partite sets. E.g. here are $K_{3,3}$ and $K_{2,2,2}$:





Graphs that are bipartite/2-colourable form an important subclass of graphs that show up in lots of applications – we'll see some later. For this reason, I'd like to prove an equivalent condition for bipartiteness:

THEOREM 6.7. *A graph is bipartite if and only if it contains no odd-length cycles.*

EXAMPLE 6.8. This is clearly a necessary condition for bipartiteness — a bipartite graph is 2-colourable by definition, and in a 2-colouring, the vertices along any cycle would have to alternate red-blue-red-blue. Therefore the cycle could not have odd length.

The proof will be our first usage of the notion of a spanning tree – recall that if G is a connected graph, then it has a *spanning tree*, i.e. a subgraph of G that is a tree and contains *all* vertices of G.

PROOF OF THEOREM 6.7. As noted, one direction is easy. For the other, suppose G contains no odd cycles. We will assume G is connected – if not, we can apply the same argument separately to each connected component.

Since G is connected, G has a spanning tree, call it T. Choose any vertex v_0 of T. By one of our theorems about trees, for each vertex v of T, there is a unique path in T from v to v_0; furthermore we can assign a direction arrow to each edge so that for all v, the unique path from v to v_0 is the unique path starting at v that points against the arrows. We colour each vertex v as follows: red if the unique path from v_0 to v has even length, and blue if it has odd length. We have now coloured all vertices of G, since T is a spanning tree.

We next show that we have given a valid colouring of G. We must check that for every edge of G, the endpoints consist of one red vertex and one blue vertex. First, we check this for edges in T, then for edges not in T.

Let e be an edge in T. The edge was assigned a direction arrow, "pointing away from v_0". Let v_1 be the endpoint of e *into which* the arrow points, and v_2 the other endpoint. Then the path from v_1 to v_0 starts with the edge e — by uniqueness, it then proceeds along the unique path from v_2 to v_0. Thus these paths differ in length by exactly 1, so they have different colours.

Next, let e be an edge not in T, and let v_1, v_2 be its endpoints. Suppose for contradiction they are the same colour. Consider the unique path P in T from v_1 to v_2. Since P contains only edges in T, we know that along P, the colours alternate. The two endpoints are the same colour, so P has an even number of edges. Adding in the edge e gives a cycle with an odd number of edges, a contradiction. Thus v_1 and v_2 are different colours. Since this argument works for any edge of G not in T, we conclude that we have given a valid 2-colouring of G. □

EXAMPLE 6.9. The Kneser graph $\mathrm{Kn}(n, k)$ is defined as follows. The vertices of $\mathrm{Kn}(n, k)$ are the k-element subsets of $[n]$. Two vertices are connected by an edge if they are disjoint.

EXERCISE 6.10. Draw $\mathrm{Kn}(5, 2)$. What does $\mathrm{Kn}(2k, k)$ look like? How many edges does $\mathrm{Kn}(n, k)$ have?

EXERCISE 6.11. Show $\mathrm{Kn}(n, k)$ can be coloured with $n - 2k + 2$ colours.

Kneser (1956) conjectured that $\chi(\mathrm{Kn}(n, k)) = n - 2k + 2$. This conjecture remained open for 20 years before Lovász found a proof using a theorem from *topology*. Let me sketch the proof in the case of $\mathrm{Kn}(5, 2)$. (Note that we have already showed this graph is not bipartite!)

PROOF SKETCH, NOT EXAMINABLE. Suppose $\mathrm{Kn}(5, 2)$ has a 2-colouring. Position the elements of $[5]$ on a sphere, with no three lying on a great circle. I define three subsets of the sphere, which possibly overlap. The subset R consists of all points P such that the hemisphere centered at P contains two elements of [5] with the corresponding vertex of $\mathrm{Kn}(5,2)$ coloured red. The subset B consists of all points P such that the hemisphere centered at P contains two elements of [5] with the corresponding vertex of $\mathrm{Kn}(5,2)$ coloured blue. The subset U consists of all points of the sphere not in R or B. Note that the hemisphere centered at a point of U contains at most 1 element of [5]. We now quote:

Theorem 6.12 (Borsuk-Ulam). *If a sphere is expressed as a union of three overlapping (open or closed) sets, then one of these sets contains a pair of antipodal points.*

Thus R, B, or U contains a pair of antipodal points. If U contains a pair of antipodal points, then 3 elements of [5] are on the complementary great circle, contradicting our assumption. Thus R or B contains a pair of antipodal points. Suppose it's R. Then I can find two elements of [5] in one hemisphere whose vertex is red, and two elements of [5] in the opposite hemisphere whose vertex is red. But these 2-element subsets of [5] must be disjoint, a contradiction. □

7. Not examinable: $\mathrm{ind}_V(\mathrm{Kn}(n,k))$ and the Erdős-Ko-Rado Theorem

We want to find an independent set of vertices in $\mathrm{Kn}(n,k)$ that is as large as possible. This is a collection of k-element subsets of $[n]$, no two of which are disjoint. Equivalently, we have n people, and we want to form committees, each of which has k people. A person is allowed to be on multiple committees, and we impose the rule that **any two committees must have a common member**. What is the maximum number of (distinct) committees can we form this way? (Clearly the answer is at most $\binom{n}{k}$.)

Example 7.1. If $n < 2k$, then the condition is trivial — any two k-element subsets of $[n]$ intersect — so the answer is just $\binom{n}{k}$.

Example 7.2. If $k = 2$ and $n = 4$, here are two sets of committees:

$$\{\{1,2\},\{1,3\},\{1,4\}\} \qquad\qquad \{\{1,2\},\{1,3\},\{2,3\}\}.$$

You can convince yourself that you can't add any elements to either of these. With slightly more thought, you can convince yourself that any 4 2-element subsets of [4] will have a disjoint pair. So 3 is the answer when $k = 2$ and $n = 4$.

Example 7.3. Extrapolating from the first set of committees above, we can just put one (busy) person in every committee — then the condition is guaranteed, and we can make $\binom{n-1}{k-1}$ committees. But can we do better?

Theorem 7.4 (Erdös-Ko-Rado 1938, proof due to Katona 1972). *This is the best we can do, i.e.* $\mathrm{ind}_V(\mathrm{Kn}(n,k)) = \binom{n-1}{k-1}$ *if* $n \geq 2k$.

Proof. It is not at all obvious how to approach this problem. It just seems like there are too many possibilities. Indeed, the original proof, which was inductive, was much more complicated than the one I'm going to present.

Suppose we have formed committees $A_1, \dots, A_r \subseteq [n]$. If we seat the n people around a circular table, we can ask ourselves, "Is committee A_i together?" ("Together" means consecutively seated.) For each seating, we count the number of committees that are together, then add that number up over all possible seatings. (We'll count two seatings as the same if they differ by rotation — there are $(n-1)!$

seatings altogether.) Let's call this sum X. It is not very well-motivated yet, but I will compute it in two ways.

Way 1 — consider each committee separately: Choose a committee A_i. In how many seatings is A_i together? There are $k!$ ways to order the members of A_i, then $(n-k)!$ ways to order everyone else. This gives a contribution of $k!(n-k)!$ from committee A_i, and adding up over all committees we get $X = r \cdot k!(n-k)!$.

Way 2 — consider each seating separately: If we choose a seating, let us count how many committees can possibly be seated together. Since any two committees have to overlap, they can't be seated at very different parts of the table — in particular, they must be shifted from each other by $< k$ elements. That is, there are at most k committees together in any given seating. Summing over the $(n-1)!$ seatings, we have $X \leq k \cdot (n-1)!$.

Putting it together, we have:

$$r \cdot k!(n-k)! = X \leq k \cdot (n-1)!$$
$$r \leq \frac{k \cdot (n-1)!}{k!(n-k)!} = \binom{n-1}{k-1}.$$

□

8. Graph traversal problems: Eulerian Graphs

An problem often credited as the origin of graph theory is the problem of the bridges of Königsberg. Is it possible to go on a walk in Konigsberg that crosses each bridge exactly once? The question and its solution were presented by Euler.

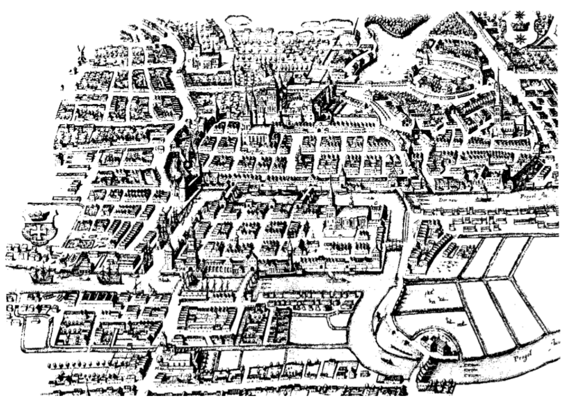
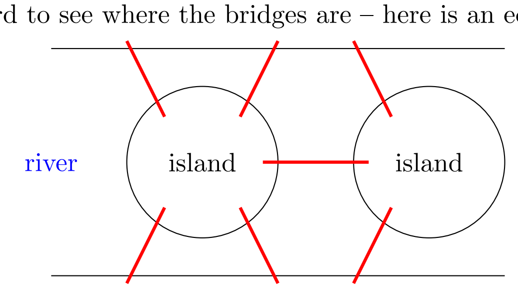
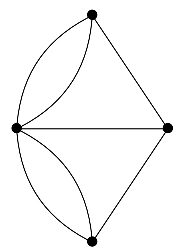


Fig. 1.8.1. The bridges of Königsberg (anno 1736)

It is a little bit hard to see where the bridges are – here is an equivalent picture:



This is equivalent to walking on the following **non-simple** graph:



(Note: In this section we can choose to allow graphs with multiple edges between the same pair of vertices, and everything works out.)

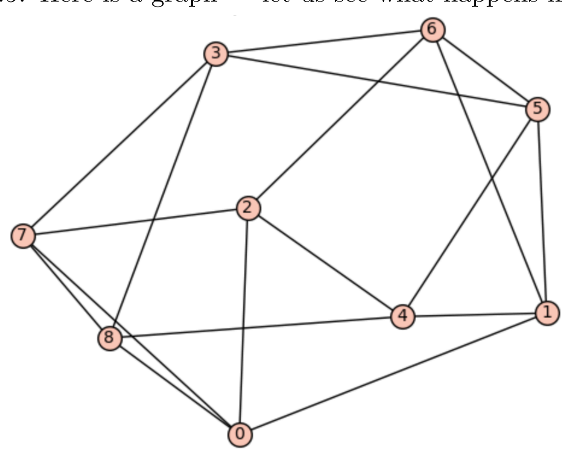
Definition 8.1. An *Eulerian circuit* of a graph G is a closed walk (ends where it starts) that traverses each edge exactly once. If an Eulerian circuit of G exists, we say G is Eulerian.

Remark 8.2. In order for an Eulerian circuit to exist, we need every vertex to have even degree. (It also must be connected.) Note that this immediately rules out the Königsberg example! Now try drawing such a graph, and see if you can draw an Eulerian circuit.

Remark 8.3. If, as in the original example, we do not require that the circuit begins where it starts, we instead get the condition that every vertex **other than the starting and ending vertices** must have even degree. (This still rules out the Königsberg graph.)

Theorem 8.4 (Euler). *A connected graph G is Eulerian if and only if every vertex has even degree.*

Example 8.5. Here is a graph — let us see what happens if we take a walk...



We can't get "stuck" unless we reach the vertex where we started! In which case, we should be able to lengthen our circuit by taking a detour at some point.

Proof of Theorem 8.4. Suppose every vertex of G has even degree. Let W be the longest walk in G that uses no edge more than once. Let v_0 be the *last* vertex of W.

Note that W must traverse every edge incident to v_0, as otherwise we could extend W. On the other hand, v_0 has even degree, so W leaves v_0 the same number of times it arrives at v_0. To account for the last arrival, W must start at v_0.

Suppose W is not an Eulerian circuit. Then there exists an edge e of G not traversed. We claim we can find such an edge that is incident to a vertex in W. To see this, if e is not already incident to W, take a path P from either endpoint of e to a vertex in W. The last edge of P before it hits W, call it e', is not traversed in W, but is incident to a vertex in W (say at a vertex v_1). Call the other endpoint v_2.

Now, consider the path that starts at v_2, travels along e to v_1, then walks along W (passing through v_0 and continuing) to end at v_1 This is a longer walk than W, a contradiction. □

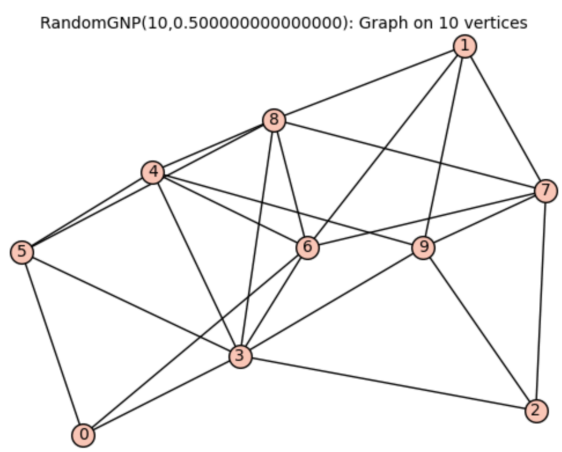
Remark 8.6. This proof also gives an algorithm for finding an Eulerian circuit. Form a closed walk W without repeating edges, just by wandering. If it does not use every edge, find a vertex with an unused incident edge, and make a new closed walk based at that vertex without using edges from W. Add the new walk to W by taking a detour at the appropriate vertex. Continue until edges are all used up.

9. Hamiltonian Graphs

Definition 9.1. A Hamiltonian cycle of a graph G is a spanning cycle in G. That is, a walk in G that is closed (ends where it started), and touches each vertex exactly once (except the start/end, which is touched twice). If G has a Hamiltonian cycle, we say G is Hamiltonian.

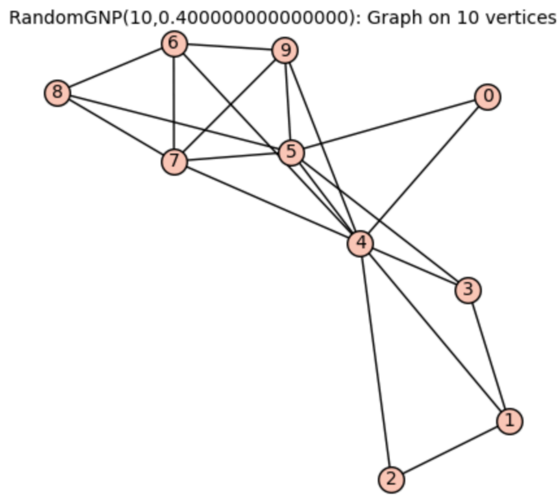
Remark 9.2. Deciding whether a connected graph is *Eulerian* was algorithmically very easy; we just had to check whether the vertex degrees were all even. Deciding whether a connected graph is *Hamiltonian* is algorithmically difficult – it is yet another example of an NP-complete problem.

Example 9.3. Here is a random graph I generated with 10 vertices:



Let us try to find a Hamiltonian cycle. How about 05468197230? In this case we were lucky.

Example 9.4. Here is another random graph I generated with 10 vertices:



Let us try to find a Hamiltonian cycle. We can immediately draw in some edges: $\{1,2\}, \{2,4\}, \{0,4\}, \{0,5\}$. So our cycle contains, in order, the vertices 5-0-4-2-1.

Now our cycle cannot contain $\{1,4\}$, so it must contain $\{1,3\}$. We have in order 5-0-4-2-1-3. We haven't made any choices so far – we were forced to this conclusion.

Now we are stuck! The next vertex in the cycle must be one of 6, 7, 8, or 9, but there is no edge from 3 to any of them. This graph is not Hamiltonian.

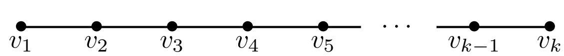
Note that Hamiltonian cycles are more likely to exist in graphs with lots of edges. For example, complete graphs have (lots of) Hamiltonian cycles. Here is the classic theorem that gives a precise version of this connection.

THEOREM 9.5 (Dirac 1952). *Suppose G is a simple connected graph with $n \geq 3$ vertices, such that every vertex has degree at least $n/2$. Then G has a Hamiltonian cycle.*

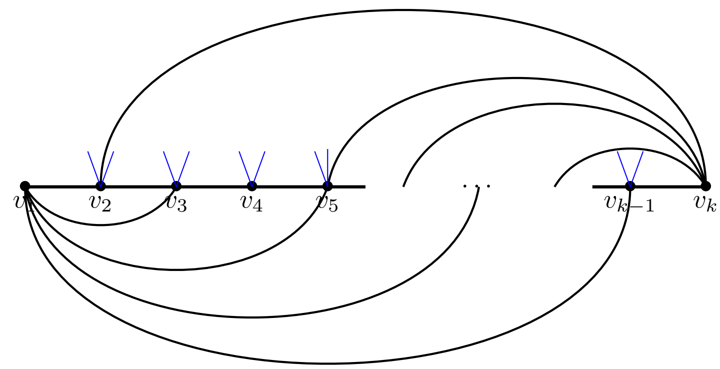
I think that the proof is one of the most elegant and clever ones we will see in the module.

PROOF. Let G be a simple graph with $n \geq 3$ vertices, such that every vertex has degree at least $n/2$.

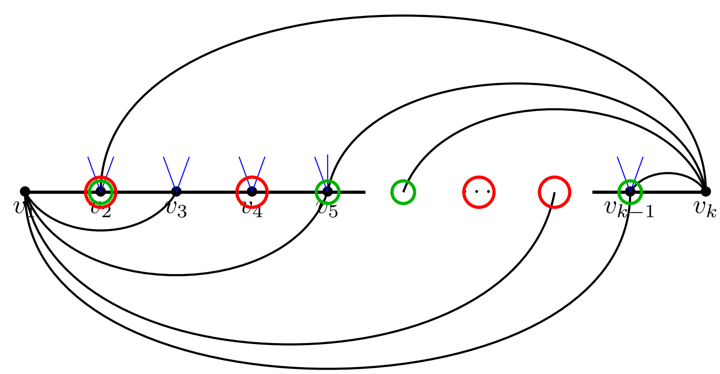
Let P be a path in G of maximum length, with vertices $v_1, v_2, \ldots, v_k$ in order. (Remember, *path* means no repeated vertices!) Draw the following:



Note $k \leq n$. By assumption, all neighbours of v_1 and v_k are in P, since otherwise we could extend P. (In theory there might be edges from the other v_is, but ignore them for now – we'll just draw some in blue to remind us they exist..) Add to the drawing:



Now we'll label some vertices. I'll put a green circle around v_i if v_i is connected to v_k. I'll also put a red circle around v_i if v_{i+1} **is connected to** v_1**.** Add to the drawing:



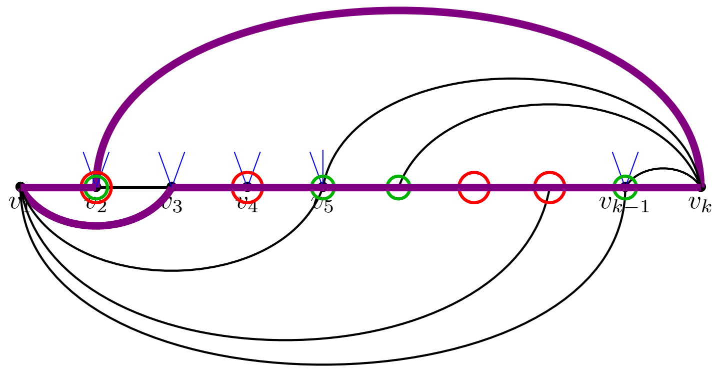
Now I claim that some vertex has both a red circle and a green circle. We establish this by counting. Since v_k has degree at least $n/2$, and every edge from v_k contributes a new green circle, there are at least $n/2 \geq k/2$ green circles. Since v_1 has degree at least $n/2$, and every edge from v_1 contributes a new red circle, there are at least $n/2 \geq k/2$ red circles. On the other hand, by construction we have *not put any circles around* v_k. That is, of the $k-1$ vertices, *more than half* have a green circle, and *more than half* have a red circle. Thus there is a vertex, say v_i, with both a green and a red circle (like v_2 in the picture).

Let C denote the cycle in G obtained by traveling:

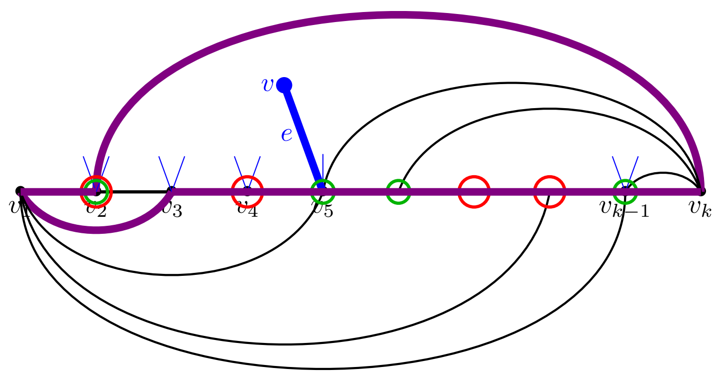
- from v_1 to v_i along P, then
- from v_i directly to v_k (along the edge guaranteed by the green circle around v_i), then
- backwards along P from v_k to v_{i+1}, then

- from v_{i+1} directly to v_1 (along the edge guaranteed by the green circle around v_i).

(This is a cycle, since we started and ended at v_1 and didn't repeat any other vertices.) Add to the drawing:



We claim C is a Hamiltonian cycle. Suppose not. Then there is a vertex *not* in C. As in the proof of Euler's theorem, we can find an edge e such that one endpoint v_j is in P, and the other, call it v, is not. Add to the drawing:

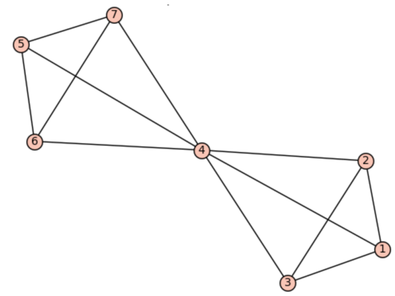


We now get a contradiction by finding a path in G that is longer than P, namely:

- Start at v and travel along e to v_j, then
- Starting at v_j, travel all the way around C, stopping *just before* hitting v_j again.

The total number of vertices in this path is $k + 1$, so it is longer than P by 1. This is a contradiction, so we conclude that C must have been a Hamiltonian cycle! □

REMARK 9.6. The example below shows we cannot replace $n/2$ with any smaller number. (Think through why this graph has no Hamiltonian cycle.)



Remark 9.7. We didn't actually need the connectedness assumption in the theorem – you can think through why it follows from the degree assumption.

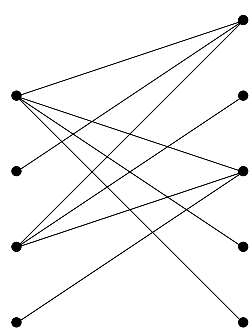
Remark 9.8. While the problem of deciding whether an arbitrary graph is Hamiltonian is NP-complete, a lot of research has gone into whether specific classes of graphs are Hamiltonian.

For example, it was proven this year that all Kneser graphs $\mathrm{Kn}(n, k)$ with $n > 2k$ are Hamiltonian, unless $(n, k) = (5, 2)$, i.e. with the exception of the Petersen graph. (Two of the authors are at Warwick!)

Another example is the Lovász Conjecture which predicts that any "vertex-transitive" graph is Hamiltonian, except for 5 specific exceptions (one of which is the Petersen graph). A graph is vertex-transitive if any two vertices are "equivalent up to symmetry"; more precisely, if for any two vertices, there is an automorphism of the graph that sends one to the other. Many special cases are known.

10. Matchings of bipartite graphs

The case of finding matchings in bipartite graphs is particularly important and well-behaved. Recall that we can draw bipartite graphs as follows:



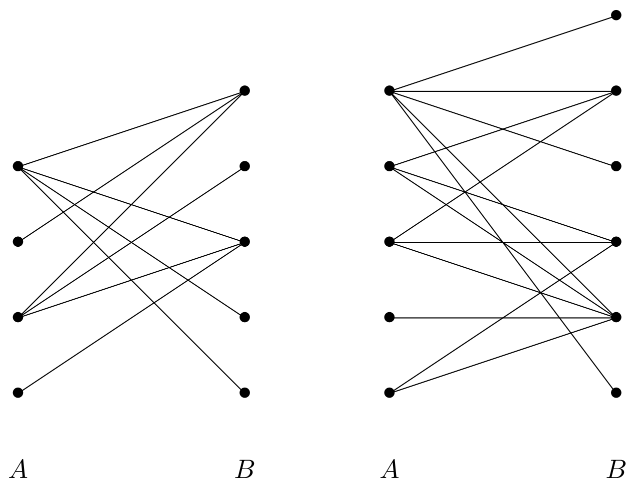
A B

Here, perhaps you have two types of object that need to be matched up, e.g. job openings and applicants. (Edges correspond to an applicant being qualified for the job, and a matching is an assignment of a qualified person to each job.)

For this section, suppose we have a graph G, whose vertices are **predivided** into partite sets A and B, so that no edges connect two vertices in A or two vertices in B. We will now discuss the search for matchings.

Remark 10.1. If $|A| < |B|$, then the largest matching we can possibly hope for is of size $|A|$, since every edge contains a vertex in A. Our main problem will be determining whether G has a *perfect* matching (only possible if $|A| = |B|$), but we might as well generalize slightly to ask if G contains a "matching of A." (If $|A| > |B|$, we just reverse the role of A and B.)

EXAMPLE 10.2. Let's see if these graphs have matchings of A:



Some examples of things that could prevent a matching of A:

- There are two degree-1 vertices in A with a (single) common neighbour.
- There are three vertices in A with only two collective neighbours.
- There are four vertices in A with only three collective neighbours. (This happened in the previous example.)

Extrapolating gives *Hall's condition*: In order to have a matching of A, we must have that for every subset $S \subseteq A$, we have $|N_G(S)| \geq |S|$ (where $N_G(S)$ is the set of all neighbours of all elements of S).

"Hall's Theorem" says this is a necessary and sufficient condition:

THEOREM 10.3 (Hall 1935). *If G satisfies Hall's condition, then there is a matching of A.*

PROOF. We use induction on $|A|$. If $|A| = 1$ the theorem is true.

Let $|A| > 1$, and suppose the theorem holds for smaller values of $|A|$. There are two cases:

(1) A stronger version of Hall's condition holds: For every proper nonempty set $S \subseteq A$, we have $|N_G(S)| \geq |S| + 1$, or
(2) There exists a proper nonempty subset $S \subseteq A$ such that $|N_G(S)| = |S|$.

In case (1), pick an edge e (say from a to b), and delete e, along with a and b (and all edges incident to them). In the resulting graph G', $|A|$ has dropped by one, and Hall's condition is still satisfied — the size of any neighbourhood has dropped by at most 1, i.e. for any $S \subseteq A \setminus \{a\}$ we have

$$|N_{G'}(S)| \geq |N_G(S)| - 1 \geq (|S| + 1) - 1.$$

Thus the inductive hypothesis guarantees a matching of $A \setminus \{a\}$ in G', and adding a, b, and e back in gives a matching of A in G.

In case (2), let $A' \subseteq A$ be such that $|N_G(A')| = |A'|$. We will try to separately find a matching of A' and a matching of $A \setminus A'$ that do not share any vertices in B.

Since $|A'| < |A|$, the inductive hypothesis implies that there is a matching of A'. The endpoints in B of this matching are a subset of $N_G(A')$ with $|A'|$ elements, i.e. they are *precisely* $N_G(A')$.

Now delete A' and $N_G(A')$ from G, say resulting in a bipartite graph G'' with partite sets $A \setminus A'$ and $B \setminus N_G(A')$. We claim that G'' satisfies Hall's condition. Let $S \subseteq A \setminus A'$. Then

$$\begin{aligned} |N_{G''}(S)| &= |N_G(S \cup A')| - |N_G(A')| \\ &\geq |S \cup A'| - |N_G(A')| \\ &= |S| + |A'| - |N_G(A')| \\ &= |S|\,. \end{aligned}$$

Thus G'' satisfies Hall's condition, so by the inductive hypothesis, there is a matching of $A \setminus A'$ in G''. Combining the matching of A' with the matching of $A \setminus A'$ yields a matching of A in G. □

REMARK 10.4. In fact, it is possible to do better, and actually find a maximal matching (even if there is no matching of A) by essentially the same proof. We won't go into it because it is slightly messy to write down.

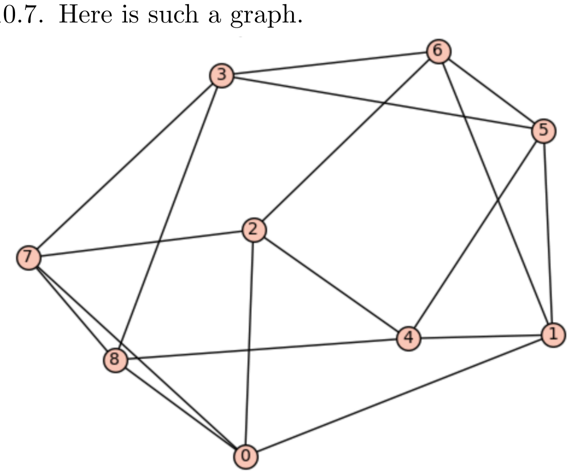
Hall's Theorem is one of the most-used theorems in graph theory. There are also lots of interesting generalizations of the bipartite matching problem as we have stated it, e.g. instead of having pairs be compatible/incompatible, each member of A may have an ordered list of preferences in B. (And many other variations that show up in various applications.)

Hall's Theorem is often applied in pretty unintuitive ways, where the structure of a bipartite graph is not obvious! To illustrate this, let's prove a very old theorem (whose original proof was longer).

THEOREM 10.5 (Petersen 1891). *Let $G = (V, E)$ be a* regular *graph of positive even degree $2k$. Then G has a spanning subgraph that is a union of disjoint cycles.*

REMARK 10.6. This is a weaker condition than being Hamiltonian.

EXAMPLE 10.7. Here is such a graph.



We can take as our cycles 356 and 014278. (Or 1635 and 24807... Can you find other ways to do it?)

Note that there are no bipartite graphs in sight. The place where Hall's Theorem gets used is in the following corollary, which you'll prove on the next Assignment:

Corollary 10.8 (Assignment 4). *A regular bipartite graph has* $|A| = |B|$ *and admits a perfect matching.*

Proof of Theorem 10.5. If G is not connected, we'll do it for each connected component separately — so assume G is connected. Let $V = \{v_1, \ldots, v_n\}$. By Euler's Theorem (Theorem 8.4), G has an Eulerian circuit W. We define a bipartite graph G' by letting $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$, with an edge from a_i to b_j if there is a step in W from v_i to v_j. (Note there is never an edge from a_i to b_i.)

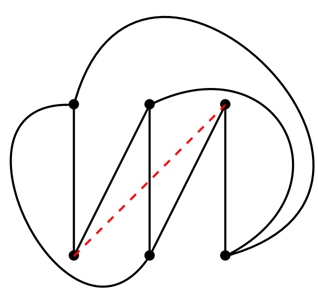
Then G' is k-regular, since W leaves (and arrives at) each vertex of G exactly k times. Thus by the previous corollary, G' admits a perfect matching M. The corresponding subgraph of G (obtained by translating the edges of M back into edges of G) has degree exactly two at every vertex of G. Thus it is a collection of disjoint cycles that hits all vertices. □

11. Planar Graphs

We now turn our attention to another important class of graphs:

Definition 11.1. The *crossing number* cross(G) of a graph G is the fewest number of edge crossings required to draw G in the plane. (Edges are not allowed to pass through vertices other than their endpoints.) A graph with crossing number zero is called a *planar graph*. So, a planar graph has a drawing in the plane with no edge crossings.

Example 11.2. It is a very classical riddle ("three houses, three utilities") that $K_{3,3}$ is not planar.



Remark 11.3. The crossing number is famously difficult to calculate, even for very straightforward graphs. For example, it is known that

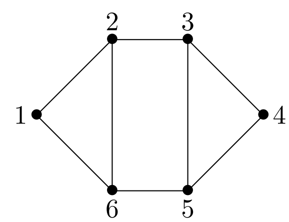
$$\operatorname{cross}(K_n) \le \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor,$$

and it is conjectured that the inequality is an equality, but this has been an open problem for about 60 years and the answer is still not known. Similarly it is known that

$$\operatorname{cross} K_{a,b} \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor,$$

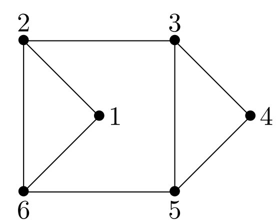
and again this is thought to be an equality, but it is not known. (This is "Turán's brick factory problem.")

Example 11.4. Here is a drawing of a planar graph.



It has 4 *faces*: two triangular faces, a rectangular face, and a hexagonal face (the outside).

Example 11.5. Here is another drawing of the same graph:



It has two triangular faces and two pentagonal faces.

Remark 11.6. As the two examples above show, we need to be careful with our language here. For example, we cannot talk about faces of a planar graph — we can only talk about faces of a specific *drawing* of a planar graph.

Remark 11.7. In the two examples, the number of faces is the same, four. The total number of sides (16) of all faces is also the same — but this amount is just double the number of edges!

Let V, E, F be the sets of vertices, edges, and faces in a drawing of a connected planar graph G. There is a famous relationship between the numbers $|V|, |E|, |F|$. Some observations:

- If we add an edge between two vertices, we will always gain one face. ($|V|$ stays constant, $|F| - |E|$ stays constant.)
- If we add a vertex to the middle of an edge, we don't change the number of faces. ($|F|$ constant, $|V| - |E|$ constant.)

Theorem 11.8 (Euler's Formula (Euler/Cauchy/etc, 1750-1811)). *For a drawing of a connected planar graph, $|V| - |E| + |F| = 2$.*

Here is a perhaps-intuitive fact:

Lemma 11.9. *Trees are planar.*

Remark 11.10. There are quite a few subtleties involved in being really truly rigorous about planar graphs. (Even the definition is imprecise as stated.) In order to get into the spirit of the combinatorics, we will ignore the subtleties.

Proof sketch of lemma. We use induction on the number of vertices. Clearly the 1-vertex tree is planar. We know that a tree G contains a leaf — remove it, and by assumption we get a planar graph; draw that graph, then draw the leaf back in without hitting the rest of the graph[1] to get a drawing of G. $\square$

Proof of Euler's Formula. Let G be a connected planar graph. Pick a planar drawing of G, with vertex/edge/face sets V, E, F. Let T be a spanning tree of G — we automatically have a planar drawing of T, just by deleting all edges of G not in T from our drawing. The drawing of T has 1 face, since the boundary of any (non-outer) face contains a cycle, and T has no cycles.

Now we add back in the edges of G not in T, one by one. Since T has $|V| - 1$ edges, there are $|E| - (|V| - 1)$ edges to add back in. Let $G_0 = T$, and we get a sequence of graphs

$$G_1, \ldots, G_{|E|-(|V|-2)}, G_{|E|-(|V|-1)} = G.$$

Call the ith edge we add this way e_i. When we add e_i to G_{i-1}, it divides a face of G_{i-1} into two faces in G_i. (As you walk down e_i in G_i, there is a face on your left and a face on your right. These can't be part of the same face, since if they were, then that face would separate the two endpoints of e_i into two different components of the graph, whereas we know G_{i-1} is connected.) Thus G_i has one more face than G_{i-1}. Since T has 1 face, we conclude that G has $1 + |E| - (|V| - 1)$ faces. That is, $F = 2 + |E| - |V|$, or $|V| - |E| + |F| = 2$. $\square$

Here is a nice corollary:

Corollary 11.11. *A simple connected planar graph with $n \geq 3$ vertices has at most $3n - 6$ edges.*

Proof. In drawing of a simple connected graph with $n \geq 3$ vertices, every face contains has at least three sides. The total number of sides, added up over all faces, is thus at least $3f$. On the other hand, it is *exactly* $2e$. (Remark: If both sides of an edge are in the same face, we should count that edge twice in order for the last sentence to be true.) Thus $2e \geq 3f$. Now

$$\begin{aligned} 2 = n - e + f \leq n - e + \frac{2e}{3} &= \frac{3n - e}{3} \\ 6 &\leq 3n - e \\ e &\leq 3n - 6. \end{aligned}$$

$\square$

Remark 11.12. The proof also showed that a simple connected planar graph has exactly $3n - 6$ edges if and only if in one (and hence every) drawing, all faces are triangles. Given a drawing without this property, one can always add edges (e.g. dividing a rectangle) until all faces are triangles.

[1] There is a subtlety here, involving the fact that the drawing might involving some terrible fractal edges or something – you should believe me when I say that we don't need to worry about such things.

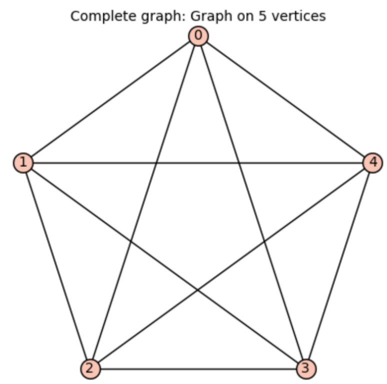
Corollary 11.13. *A simple planar graph has a vertex with degree at most 5.*

Proof. The average degree of a vertex is

$$\frac{1}{n}\sum_{v\in V}\deg(v)=\frac{1}{n}(2\,|E|)\leq\frac{6n-12}{n}<6.$$

Thus there is a vertex v with $\deg(v)<6$. □

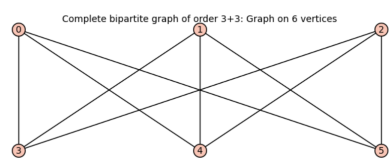
Corollary 11.14. *K_5 is not planar.*



Proof. K_5 has $10>3\cdot 5-6$ edges. □

In fact, we can solve the 3-houses-3-utilities problem in a similar way.

Corollary 11.15. *$K_{3,3}$ is not planar.*



Proof. As a bipartite graph, $K_{3,3}$ has no odd cycles, in particular no triangles. Thus if a drawing existed, each face would need to have at least *four* sides, so $2e\geq 4f$. The equations above become

$$2=n-e+f\leq n-e+\frac{2e}{4}=\frac{2n-e}{2}$$
$$4\leq 2n-e.$$

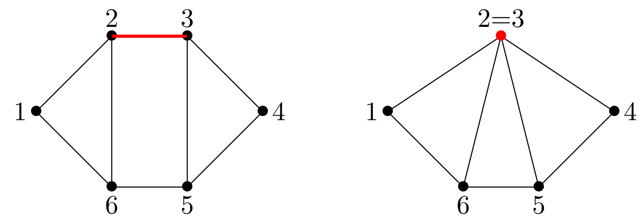
But we have $n=6$ and $e=9$, so this reads $4\leq 12-9$, a contradiction. □

12. The Kuratowski-Wagner Theorem

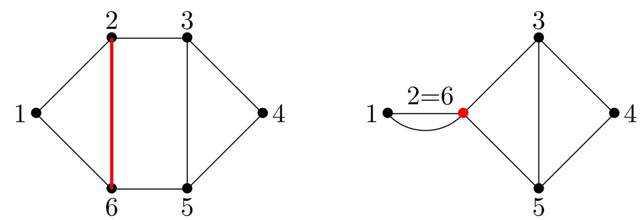
The main theorem of this section is a kind of converse to the last two examples in the previous section — it says that if a graph is *not* planar, then it must have K_5 or $K_{3,3}$ inside it in some way. We need to introduce some language to state this.

Definition 12.1. Given a graph G and an edge e of G, we can *contract* e; this means that we remove e and merge its two endpoints. We can also *delete* e (leaving its endpoints alone).

EXAMPLE 12.2. Here is a graph, and the result of the contraction of the red edge:



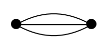
Here is another contraction of the same graph:

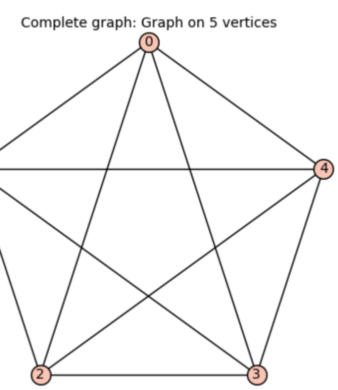


Note that the result is not a simple graph. We could also contract one of the left edges in the last graph to get a loop...

DEFINITION 12.3. Given a graph G and a vertex v, we can *delete* v which also includes deleting all edges incident to v.

DEFINITION 12.4. A graph H is a *minor* of a graph G if H can be obtained from G by a sequence of edge contractions, edge deletions, and vertex deletions.

EXAMPLE 12.5. Does K_5 have the graph  as a minor?



(Delete 2, leaving K_4, then delete $\{0, 3\}$, then contract $\{0, 4\}$ and $\{3, 4\}$.)

Suppose H is a minor of G, and G is planar. I claim so is H — just "do the edge contraction in a drawing". Put differently, if H is a minor of G, and H is not planar, then neither is G. In particular, by Corollaries 11.14 and 11.15: **if G is a graph that has $K_{3,3}$ or K_5 as a minor, then G is *not* planar.** Our next big theorem is the converse:

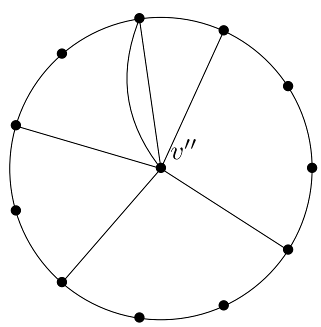
THEOREM 12.6 (Kuratowski 1930, Wagner 1937). *A graph G is planar if and only if it if does not have $K_{3,3}$ or K_5 as a minor.*

Remark 12.7. It is worth appreciating what Theorem 12.6 does for us — it says that planarity, an apparently *noncombinatorial* condition (about the geometry of drawings of G), can actually be decided combinatorially.

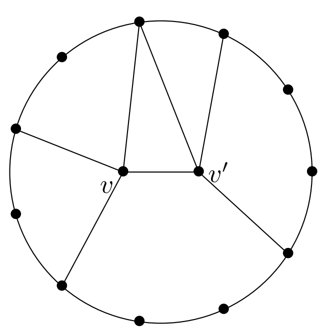
Idea of proof — not examinable. The full proof has several technical lemmas, but I will try to communicate the spirit of why, if a graph is not planar, then K_5 and $K_{3,3}$ must show up. We may work consider only simple graphs, since as we noted, multiple edges and loops do not affect planarity.

We induct on the number of vertices, with base case $n = 4$. A simple graph on 4 vertices is a subgraph of K_4, hence is planar. Now suppose the theorem holds for $|V| \leq n-1$, and suppose G has n vertices, and does not have $K_{3,3}$ or K_5 as a minor.

Pick an edge $\{v, v'\}$ and contract it, calling the resulting vertex v''. The resulting graph (which is not necessarily simple) has $n-1$ vertices and does not have K_5 or $K_{3,3}$ as a minor (since a minor of a minor is a minor), so is planar. Draw it. The vertex v'' has some collection of incident edges, dividing some collection of faces: that is, if we "zoom in near v''", the picture must look "something like" this[2]:

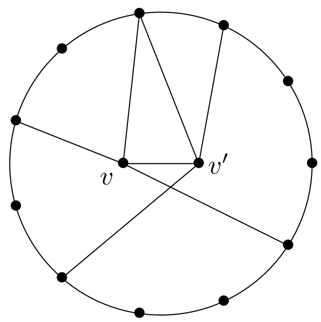


(I've drawn only the faces that touch v''.) Now in order to turn this into a drawing of G, we need to split v'' back up into v and v', appropriately assigning the edges incident to v'' to v and v'. Something like this:

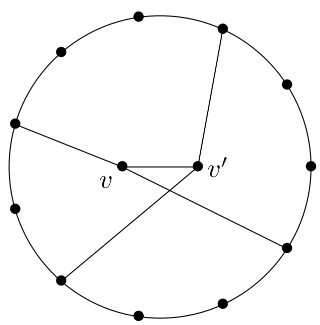


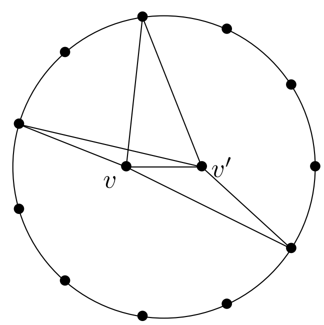
This is possible *provided* the vertices that need to connect to v lie consecutively along the outer cycle. Otherwise we might have something like this:

[2]This statement might seem dubious – in fact some small arguments are needed here, which I omit.



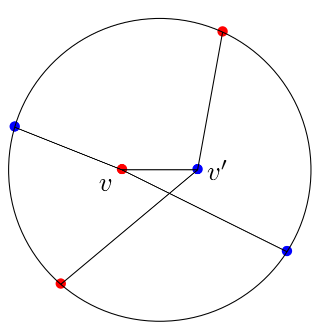
There are exactly two "basic" ways in which this could go wrong:

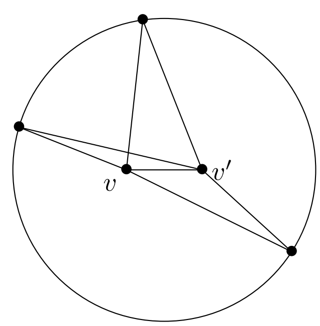




On the left is the case where v' needs to reach two different segments of the outer cycle that are separated by edges from v. You might think that this is the only thing that could go wrong — but in fact there's one more, on the right.

But both of these possibilities are ruled out! The left one contains a $K_{3,3}$ as a minor, the right one contains a K_5 :





□

Remark 12.8. Omitted from lecture, not examinable. The graph minor theorem, proved by Robertson and Seymour in 2004, says that for any graph property P (in this case, planarity) that is preserved under taking minors, there exists an analogue of the Kuratowski-Wagner Theorem — that is, there is a finite collection of graphs $H_1, \ldots, H_r$ such that G has property P if and only if it does not have $H_1, \ldots, H_r$ as a minor. This is a deep theorem, proved over 20 years, involving 20 papers totaling over 500 pages, and it has a huge impact on modern graph theory research.

13. The 4/5/6 colour theorems

Recall that when I first introduced graphs, I talked about the 4-colour theorem:

THEOREM 13.1 (Appel-Haken etc.). *If G is planar, then $\chi(G) \leq 4$.*

As mentioned, all known proofs are computer-assisted. We will instead prove the *5* colour theorem, but let's warm up with something easier — the 6 colour theorem:

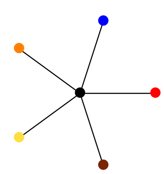
THEOREM 13.2. *If G is planar, then $\chi(G) \leq 6$.*

PROOF. Induct on the number of vertices, with base case $n = 1$. Suppose the theorem holds for planar graphs with $n-1$ vertices. Let G be a planar graph with n vertices. By Corollary 11.13, G has a vertex v of degree at most 5. Delete v and 6-colour the remaining vertices by induction. Since v has degree at most 5, we can colour it. □

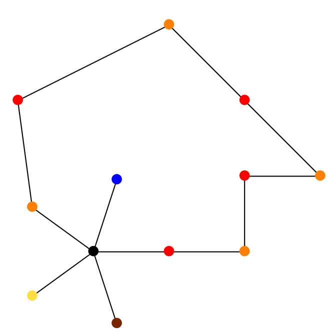
THEOREM 13.3 (Heawood 1890). *If G is planar, then $\chi(G) \leq 5$.*

PROOF. Induct on the number of vertices, with base case $n = 1$. Suppose the theorem holds for planar graphs with $n-1$ vertices. Let G be a planar graph with n vertices. Again, by Corollary 11.13, G has a vertex v of degree at most 5. Delete v and 5-colour the remaining vertices by induction.

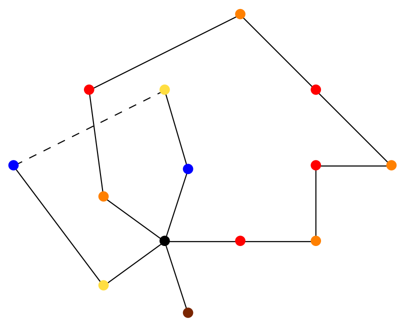
If $\deg(v) < 5$, we can colour v, so assume $\deg(v) = 5$. If the five neighbours of v do not all have different colours, we can colour v, so assume they do. We have:



Consider *all* red and orange vertices and edges between them. These form a subgraph. If the red and orange vertices shown in the picture above are in *different* connected components of this subgraph, then we can flip red and orange in one of those two connected components, getting a valid colouring where only 4 colours are used to colour the neighbours of v — then we can colour v and are done. If not, there is a path (alternating orange-red) connecting these two vertices:



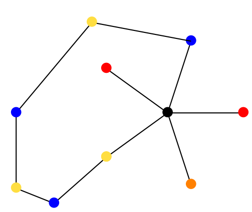
Similarly, consider *all* blue and yellow vertices and edges between them. These form a subgraph. If the blue and yellow vertices shown in the picture above are in different connected components of this subgraph, then we can flip blue with yellow in one of those two connected components, getting a valid colouring where only 4 colours are used to colour the neighbours of v — then we can colour v and are done. If not, there is a path (alternating blue-yellow) connecting these two vertices. But such a path would have to cross the red-orange path, a contradiction! We conclude that $\chi(G) \leq 5$.



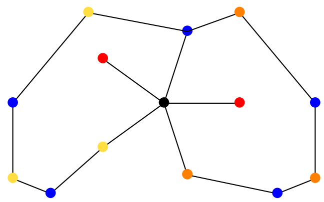
□

Remark 13.4. Let's see why Kempe thought the above argument proved the 4-colour theorem. Suppose we tried to use the same strategy to prove the 4-colour theorem by induction. As before, G has a vertex v with degree at most 5. Delete v and 4-colour the remaining vertices by induction. If $\deg(v) < 5$, it is not hard to check that the proof we just gave of the 5-colour theorem gives a 4-colouring of G — this is basically because there was a vertex in the above argument that was never used. If $\deg(v) = 5$ and its neighbours do not exhaust all four colours, we also have a valid 4-colouring. There are, up to symmetry, two remaining possibilities for what the neighbourhood of v looks like:

For the left case, you can check that the 5-colouring argument again works without modification to give a valid 4-colouring. For the right case, we can flip the blue vertex to yellow unless there is an alternating path like this:

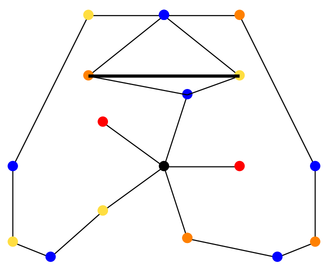


Similarly we can flip blue to yellow unless there is also an alternating path like this:



Now, the left red vertex is separated from the orange vertex, so we can flip red/orange on the connected component of the red/orange subgraph containing the left red vertex. Similarly, we can flip red/yellow on the connected component of the red/yellow subgraph containing the right red vertex. As a result, we can colour v red.

Where does this go wrong? The picture is misleading — the two alternating paths we constructed might intersect in a blue vertex, e.g.:



Now if we go through the supposed proof, we may have to flip both endpoints of the thick edge to red, creating a problem.

14. Some Ramsey theory

Ramsey theory is a branch of combinatorics that deals with what kinds of structures are guaranteed to exist in sufficiently large sets. Here is an example of such a structure, from Assignment 4:

Example 14.1. Show that in any group of at least six people, one can find either three people who all know each other, or three people who all do not know each other.

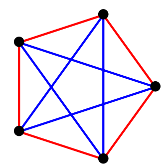
Hopefully thinking about this problem on the assignment has convinced you that it is not a totally obvious fact! Here are a couple of translations to graph theory:

Example 14.2. Every graph with at least 6 vertices has a triangle or a 3-vertex independent set. (Here the edges correspond to pairs of people who know each other.)

Example 14.3. colour the edges of K_6 with two colours, say red and blue. Then there is a red triangle or a blue triangle. (Here the red edges are pairs of people who know each other.)

I'll reproduce a solution to this problem here. Let us search for red triangles. Pick a vertex v with at least three red edges from it — we can assume, after possibly flipping colours, such a vertex exists. Let a, b, c be the endpoints of these three red edges. What are the colours of the edges connecting a, b, c? If they are all blue, we have, of course, a blue triangle. If not, one of them is red — then its endpoints, together with v, form a red triangle.

REMARK 14.4. Is it true if we have only five people? No:



Is it true for larger groups of people than 6? Of course.

The foundational theorem of Ramsey theory is a generalization of our example:

THEOREM 14.5 (Ramsey 1930). *Let $r, s \geq 2$. Then for sufficiently large n, every 2-colouring of the edges of K_n contains a red K_r or a blue K_s.*

DEFINITION 14.6. The minimal such n is called the *Ramsey number* $R(r, s)$.

REMARK 14.7. We proved $R(3, 3) = 6$. (We first proved $R(3, 3) \leq 6$, then constructed an example that shows $R(3, 3) > 5$.)

REMARK 14.8. In fact there are more general Ramsey numbers e.g. $R(3, 3, 4)$, the minimum n such that every 3-colouring of K_n has a red K_3, a blue K_3, or a green K_4.

The proof of Ramsey's Theorem is essentially a generalization of the proof we did that $R(3, 3) \leq 6$.

PROOF OF THEOREM 14.5. Just as before, we will find a vertex with either a large number of red edges — enough that, by induction, the other endpoints guarantee a red K_{k-1} or a blue K_ℓ — or a large number of blue edges — enough that, by induction, the other endpoints guarantee a red K_k or a blue $K_{\ell-1}$.

We induct on $k + \ell$, where the base cases are $k = 2$ (and ℓ arbitrary) and $\ell = 2$ (and k arbitrary). (To make sure you understand the definitions correctly, think through what $R(k, 2)$ and $R(2, \ell)$ are.) Suppose $k, \ell > 2$ and that the theorem holds for $(k - 1, \ell)$ and $(k, \ell - 1)$. (That is, the exist Ramsey numbers $R(k - 1, \ell)$ and $R(k, \ell - 1)$ satisfying the property in the theorem.) Consider the complete graph $K_{R(k-1,\ell)+R(k,\ell-1)}$. Let v_0 be a vertex. We must have $R(k - 1, \ell)$ red edges or $R(k, \ell - 1)$ blue edges, since otherwise we would have at most

$$(R(k - 1, \ell) - 1) + (R(k, \ell - 1) - 1) = R(k - 1, \ell) + R(k, \ell - 1) - 2$$

edges incident to v_0, whereas we know that we actually have $\deg(v_0) = R(k - 1, \ell) + R(k, \ell - 1) - 1$ such edges.

If we have $R(k - 1, \ell)$ red edges: Among the endpoints of these edges, we have a red K_{k-1} — giving, with v_0, a red K_k — or a blue K_ℓ.

If we have $R(k, \ell - 1)$ blue edges: Among the endpoints of these edges, we have a red K_k, or a blue $K_{\ell-1}$ — giving, with v_0, a blue K_ℓ.

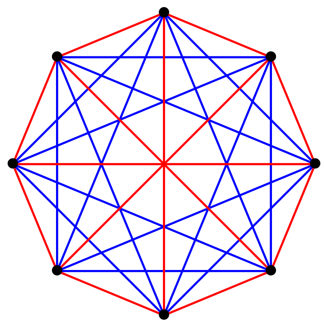
In either case, we have a red K_k or a blue K_ℓ, and are done. □

REMARK 14.9. We actually showed $R(k, \ell) \leq R(k - 1, \ell) + R(k, \ell - 1)$. In fact, this recursive structure looks very similar to the recursion for binomial coefficients, and you can easily use it to prove by induction:

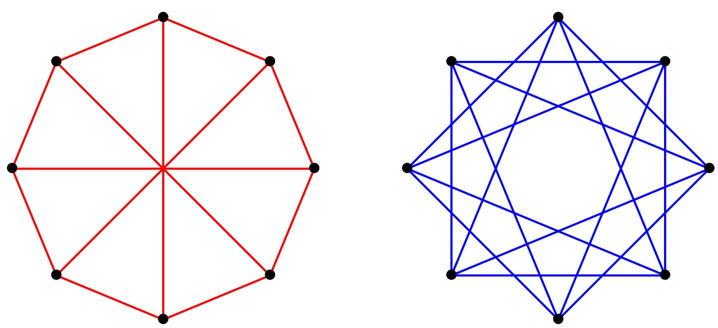
COROLLARY 14.10. $R(k,\ell) \le \binom{k+\ell-2}{k-1}$.

The upper bound of Corollary 14.10 gives $R(3,3) \le \binom{4}{2} = 6$, and we know that actually $R(3,3) = 6$. Let's quickly see that we are not always so lucky. The upper bound gives $R(3,4) \le \binom{5}{2} = 10$, but we now show that actually $R(3,4) = 9$.

EXAMPLE 14.11. To show $R(3,4) = 9$, we need to show two things. First, that it is possible to 2-colour the edges of K_8 so that there is no red K_3 or blue K_4. We attempt to do so, and come up with:



The red and blue edges alone look like this:



Convince yourself there are no red triangles on the left, or blue K_4s. (The four blue edges from a vertex come in two pairs, neither of which form a triangle; picking three edges must include one of the pairs...)

Next, we need to show that if the edges of K_9 are 2-coloured, there is a red K_3 or blue K_4. We can think along the same lines as in the proof of Ramsey's Theorem. If we can find a vertex v_0 with 4 red edges coming out of it, say to vertices v_1, v_2, v_3, v_4, then either all 6 edges between $v_1, \ldots, v_4$ are blue — giving a blue K_4 — or one is red, giving a red triangle with v_0.

Is it necessarily true that some vertex has 4 red edges? (Of course not, e.g. *all* edges could be blue...) Suppose that every vertex has at most 3 red edges. Then the total number of red edges is at most $\frac{1}{2}(3 \cdot 9) = 27/2$. Since the number of red edges is an integer, it is at most 13, leaving at least $36 - 13 = 23$ blue edges. What does this get us? Well, note that the average number of blue edges from a vertex is $\frac{1}{9}(23 \cdot 2) = 46/9 > 5$. So we at least know that there is a vertex w_0 with six blue edges coming out of it. Let's see if we can use this!

Call the other endpoints of these six edges $w_1, \ldots, w_6$. The edges between $w_1, \ldots, w_6$ are coloured red or blue. Since $R(3,3) = 6$, there is either a red triangle or a blue triangle among $w_1, \dots, w_6$. If there is a red triangle, we are done. If there is a blue triangle, then together with w_0 we have found a blue K_4.

REMARK 14.12. Computing Ramsey numbers, while not our main interest, is notoriously difficult. For example, $R(5,5)$ is somewhere between 43 and 48, but we do not know the exact value.

The next thing I'd like to do is prove a *lower* bound for $R(k,k)$, in order to showcase an extremely influential proof technique in combinatorics called the *probabilistic method*.

THEOREM 14.13 (Erdös (1947)). *For $k \geq 4$, we have $R(k,k) > 2^{k/2}$.*

PROOF. To prove this, we will show that if $n \leq 2^{k/2}$, and we select a *random* 2-edge-colouring of K_n (for each pair of edges, we flip a coin to choose its colour), then the probability of a red K_k is *less than* $1/2$ (and similarly, by symmetry, for the probability of a blue K_k). Thus there must exist colourings where neither occurs.

Let us count the total number of red K_ks among all 2-edge-colourings of K_n. For each choice of k vertices, the number of colourings in which the corresponding K_k is red is $2^{\binom{n}{2}-\binom{k}{2}}$. Thus the total number of red K_ks among all 2-edge-colourings is $\binom{n}{k} \cdot 2^{\binom{n}{2}-\binom{k}{2}}$.

We want to show that the number, M_{red}, of 2-edge-colourings that *contain* a red K_k is relatively small. The worst-case scenario would be if each of the $\binom{n}{k} \cdot 2^{\binom{n}{2}-\binom{k}{2}}$ red K_ks above appear in different colourings. Of course, they don't — we certainly do have 2-edge-colourings with multiple red K_ks — but we do get the inequality

$$M \leq \binom{n}{k} \cdot 2^{\binom{n}{2}-\binom{k}{2}}.$$

Now if $n \leq 2^{k/2}$, we get

$$\begin{aligned}
M_{red} &\leq \binom{n}{k} \cdot 2^{\binom{n}{2}-\binom{k}{2}} \\
&= \frac{n \cdot (n-1) \cdot \dots \cdot (n-(k-1))}{k!} \cdot 2^{\binom{n}{2}-\binom{k}{2}} \\
&< \frac{n^k}{k!} \cdot 2^{\binom{n}{2}-\binom{k}{2}} \\
&< \frac{n^k}{2^k} \cdot 2^{\binom{n}{2}-\binom{k}{2}} \\
&\leq \frac{2^{k^2/2}}{2^k} \cdot 2^{\binom{n}{2}-k(k-1)/2} \\
&= \frac{1}{2^{k/2}} \cdot 2^{\binom{n}{2}} \\
&< \frac{1}{2} \cdot 2^{\binom{n}{2}}.
\end{aligned}$$

By symmetry M_{blue} has the same property, so $M_{red} + M_{blue} < 2^{\binom{n}{2}}$. Thus there must exist 2-edge-colourings with no red or blue K_k.

□

REMARK 14.14. This proof is nonconstructive — and no constructive proof is known!

This says something interesting — if you want a 2-edge-colouring of K_n with no monochromatic K_k, a good way of doing it is to pick colours at random. In fact, our probability estimate above shows that if k is large, with $n \leq 2^{k/2}$, it is **very** unlikely that there is a monochromatic K_k in a random 2-edge-colouring. On the other hand, if you try to write down a 2-edge-colouring systematically, you are likely to make it "not random enough," and end up with monochromatic K_ks.

Mathematicians sometimes describe this phenomenon as the difficulty of "finding hay in a haystack"; even if most graphs satisfy some property, it may be difficult to actually write one down that you can prove satisfies the property, for any given k.

Remark 14.15. I called this a "probabilistic" argument, but it was really just a counting argument. You could instead phrase it as: "We calculated the probability that a random 2-edge-colouring has a red K_k." In more complicated arguments, such as the next one, we'll see that the language of probability really does streamline the argument.

Here are two more results in Ramsey theory that are not (explicitly) about graphs.

Theorem 14.16 (Schur's Theorem). *Suppose you have 5 (or k) colours. For sufficiently large n, if you colour the set $[n]$ with your 5 colours, then there will exist $x, y \in [n]$ such that x, y, and $x + y$ all have the same colour.*

Theorem 14.17 (Erdös-Szekeres-Klein). *Given sufficiently many noncollinear points in $\mathbb{R}^2$, there exist 7 of them that are the vertices of a* convex *7-gon. (Or replace 7 with whatever k you like.)*

15. Another probabilistic proof

If you try to create a graph with large chromatic number, you will probably end up drawing something with a large complete subgraph. In fact, for every r, Mycielski (1955) constructed a graph with chromatic number r and no triangles. (See the optional material in section 16.) Erdös proved the following generalization:

Theorem 15.1 (Erdös (1959)). *For any positive integers k and r, there exists a graph with chromatic number $> r$, with girth $> k$. (That is, no cycles of length $\leq k$.)*

This is a nonconstructive proof that uses the probabilistic method in a very beautiful way. (Note: $<$ vs $\leq$ unimportant.)

Proof. Fix k and r. We are going to take n to be very large, and we are going to pick a random simple graph G with n vertices as follows: For each pair of vertices, draw an edge between them with probability $p = \frac{1}{n^{1-\epsilon}}$, for some very small $\epsilon > 0$. This probability is just a little bit *larger* than $1/n$.

Here is the plan: It would be nice to show that if n is very large, then with nonzero probability, G will have no cycles of length $\leq k$, and will have chromatic number $> r$. However, this turns out not to be the case — in fact, it will turn out that for n very large, G will likely have large chromatic number, but probably *will* have cycles of length $\leq k$ — just not very many of them.[3] Few enough, in fact, that

[3]Why didn't we then just make p a little bit smaller? It turns out that if we decrease p, we cross into the "no short cycles" zone at the same time as we *leave* the "large chromatic number" zone. (In fact, below this threshold, as $n \to \infty$, it becomes highly likely that G is acyclic, which implies $\chi(G) \leq 2$.)

we will be able to delete a vertex from each cycle and still have enough vertices and edges to know that the chromatic number is large.

For convenience, we will call cycles of length $\leq k$ simply "short cycles." Here is the quantitative version of the statement "G probably will have only a few short cycles." More precisely:

Claim 1: *On average*, G will have $\leq \frac{1}{2}(k-2)n^{k\epsilon}$ short cycles.

Proof of Claim 1. The number of possible k-cycles is

$$\frac{n \cdot (n-1) \cdot \dots \cdot (n-(k-1))}{2k}.$$

(This is the number of orderings of k vertices, but we divide by $2k$ because we don't care about the starting point, or the order, of the cycle.) For each of these possible cycles, it exists in G with probability p^k.

Thus the average over G, weighted by probability, of the number of k-cycles in G is

$$E_k := \frac{n \cdot (n-1) \cdot \dots \cdot (n-(k-1))}{2k} p^k \leq \frac{n^k}{2} p^k = \frac{1}{2} n^{k\epsilon}.$$

(In probability, such an average is called the *expected value*.)

To finish the proof of the claim, we add up over lengths $1, \dots, k$ (ignoring 1 and 2 because our random graphs are all simple). The weighted average/expected value of the number of short cycles is

$$E := E_3 + \dots + E_k \leq \sum_{i=3}^{k} \frac{1}{2}(n^\epsilon)^i \leq \frac{1}{2} \sum_{i=3}^{k} (n^\epsilon)^k = \frac{1}{2}(k-2)n^{k\epsilon}.$$

(Note $n^\epsilon \geq 1$, so $(n^\epsilon)^k \geq (n^\epsilon)^i$.) □

We now know that G has few short cycles *on average*. We need something a bit stronger. This "average"/expected value was easy to calculate, but we are looking for not a statement about averages, but a statement about probabilities.

Claim 2: For large n, G is very unlikely to have more than $n/2$ short cycles. (This is the sort of probabilistic statement we need. As mentioned above, after we know this, we will be able to take G and delete a vertex from each cycle, and still have a large graph left over.)

Proof of Claim 2. We know that, on average, G has $\leq \frac{1}{2}(k-2)n^{k\epsilon}$ short cycles. We want an upper bound on the probability P that G has more than $n/2$ short cycles. To get an upper bound, consider the "worst-case scenario", where that average is due to only two types of graph: graphs with exactly $n/2$ short cycles, and graphs with zero short cycles. (So all of the short cycles are contributing to P as much as possible, with no redundancy. This gives as high a value of P as possible given the average number of short cycles.) In this worst case, we would have:

$$E = \frac{n}{2} \cdot P + 0 \cdot (1-P) \leq \frac{1}{2}(k-2)n^{k\epsilon}$$

$$P \leq \frac{2}{n}\left(\frac{1}{2}(k-2)n^{k\epsilon}\right) = (k-2)n^{k\epsilon-1}.$$

(This inequality holds even in our worst-case estimate, so it will certainly work in reality. Note for those who have seen probability: we just used Markov's inequality.) As ϵ is very small (in particular, we should have picked $\epsilon < 1/k$), the exponent is negative, so $P \to 0$ as $n \to \infty$. That is, for very large n, the probability of more than $n/2$ short cycles is very small. □

Next we need to make sure that if we take such a graph G (for very large n) and delete a vertex from each of the $\leq n/2$ short cycles, we get a graph H with chromatic number $> r$. How do you show that H has large chromatic number? One way is to show that $\text{ind}_V(H)$ is small — for each colour, the set of vertices of that colour is an independent set. So if $\text{ind}_V(H) < \frac{\#\text{ vertices of } H}{r}$, then $\chi(H) > r$. We know H has at least $n/2$ vertices, so it is sufficient to show that $\text{ind}_V(H) < \frac{n}{2r}$.

Note that an independent subset of H is also an independent subset of G, so it is enough to show that $\text{ind}_V(G) < \frac{n}{2r}$.

Claim 3: If n is large, then with high probability, G will satisfy $\text{ind}_V(G) < \frac{n}{2r}$.

Once we prove this claim, we will be done!

Proof of Claim 3. We actually almost did this already. In the proof of the Ramsey number lower bound, we coloured the edges of K_n with red and blue, and said that the probability of a blue K_a is less than $\binom{n}{a}/2^{\binom{a}{2}}$. (Switched notation from k to a because we already have a k in this proof.) If we had chosen edges to be red with probability p and blue with probability $1-p$, the probability of a blue K_a would be, by the same argument, $\binom{n}{a} \cdot (1-p)^{\binom{a}{2}}$. A blue K_a is the same as an independent set of size a in the subgraph G consisting of all n vertices and all red edges.

The only difference is choosing which edges are red/blue versus which edges do/don't exist – but these are completely equivalent. Thus the probability that $\text{ind}_V(G) \geq a$ is at most $\binom{n}{a} \cdot (1-p)^{\binom{a}{2}}$.

In particular, the probability that $\text{ind}_V(G) \geq \frac{n}{2r}$ is at most

$$\binom{n}{n/2r} \cdot (1-p)^{\binom{n/2r}{2}} = \binom{n}{n/2r} \cdot (1-n^{\epsilon-1})^{\binom{n/2r}{2}}.$$

It is sufficient to show that this approaches zero as $n \to \infty$, a somewhat tedious analysis exercise. Here is the argument. We have:

$$\begin{aligned}\binom{n}{n/2r} \cdot (1-n^{\epsilon-1})^{\binom{n/2r}{2}} &\leq n^{n/2r}(1-n^{\epsilon-1})^{(n/2r)(n/2r-1)/2} \\ &= \left(n(1-n^{\epsilon-1})^{(n/2r-1)/2}\right)^{n/2r}\end{aligned}$$

It is sufficient to show that the quantity inside the parentheses approaches zero as $n \to \infty$. We have

$$n(1-n^{\epsilon-1})^{(n/2r-1)/2} \leq n(e^{-n^{\epsilon-1}})^{(n/2r-1)/2},$$

due to the fact that $0 \leq 1-x \leq e^{-x}$. We then have:

$$\begin{aligned}n(e^{-n^{\epsilon-1}})^{(n/2r-1)/2} &= n \cdot \exp(-n^{\epsilon-1}(n/2r-1)/2) \\ &= n \cdot \exp\left(-\frac{n^{\epsilon}}{4r} + \frac{n^{\epsilon-1}}{2}\right) \\ &= \exp\left(\ln(n) - \frac{n^{\epsilon}}{4r} + \frac{1}{2n^{1-\epsilon}}\right).\end{aligned}$$

As $n \to \infty$, the third summand in the exponent goes to zero, and we now use the fact that n^ϵ (and $n^\epsilon/4r$) grows faster than $\ln(n)$. Thus the exponent approaches $-\infty$, so the expression approaches zero.

Putting the inequalities all together, we have shown that with high probability, if n is large, we have $\mathrm{ind}_V(G) < \frac{n}{2r}$. □

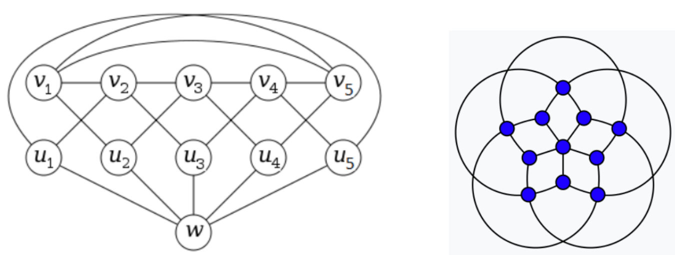
Taking n sufficiently large, we have now shown that with high probability, if we choose G according to our random scheme (i.e. each edge exists with probability $n^{\epsilon-1}$), then with high probability, G has at most $n/2$ short cycles, and deleting a vertex from each yields a graph H with no independent sets large enough to allow an r-colouring. □

16. Not examinable — Triangle-free graphs with high chromatic number: The Mycielski construction

This section was not covered in lecture. We give a constructive argument for the previous theorem in the case $k = 3$. This predates Erdős's result by a few years.

THEOREM 16.1 (Mycielski, 1955). *For any positive integer $r \geq 2$, there exists a graph M_r with chromatic number r that contains no 3-cycles.*

EXAMPLE 16.2. Here are two pictures of M_4 — you can see that it has no triangles, and you can convince yourself after a few attempts that there is no 3-colouring.



REMARK 16.3. This is the $k = 3$ case of Erdős's theorem from the last section. In 1968, Lovász found a constructive proof of the general theorem.

PROOF OF MYCIELSKI'S THEOREM. We define $M_2 = P_2$. We then define M_r recursively as follows. Suppose M_r has vertices $v_1, \ldots, v_n$. We define M_{r+1} to be the graph with vertices $v_1, \ldots, v_n, u_1, \ldots, u_n, w$ and edges:

- an edge $\{v_i, v_j\}$ if and only if $\{v_i, v_j\}$ is an edge of M_r,
- an edge $\{u_i, v_j\}$ if and only if $\{v_i, v_j\}$ is an edge of M_r (in particular, u_i is not adjacent to v_i)
- an edge $\{u_i, w\}$ for all $i = 1, \ldots, n$,
- no edges of the form $\{u_i, u_j\}$,
- no edges of the form $\{v_i, w\}$.

We now prove that M_r is triangle-free and has chromatic number r. But first, examples:

EXAMPLE 16.4. Verify that $M_3 = C_5$ and M_4 is as pictured above.

Claim 1: M_r is triangle-free. We induct on r, with easy base case $r = 2$. Suppose M_{r-1} is triangle-free. What could a triangle in M_r look like? It cannot contain w, as w is only adjacent to u_is, and there are no edges between the u_is. Similarly it cannot contain two u_is. By the inductive hypothesis, it cannot contain three v_is. The only possibility is that it consists of three vertices v_i, v_j, u_k, where v_i and v_j are adjacent (in M_r, and therefore also in M_{r-1}). Note k cannot be equal to i or j since u_k is not adjacent to v_k. But if $\{u_k, v_i\}$ and $\{u_k, v_j\}$ were edges of M_r, then by definition $\{v_k, v_i\}$ and $\{v_k, v_j\}$ would be edges of M_{r-1} – since v_i and v_j are adjacent in M_{r-1}, this contradicts the inductive hypothesis. Thus there are no triangles in M_r.

Claim 2: M_r has chromatic number r. We again induct on r, with easy base case $r = 2$. Suppose $\chi(M_{r-1}) = r - 1$. We may colour M_r with r colours by choosing a colouring of M_{r-1} with $r - 1$ colours, colouring u_i with the same colour as v_i, and colouring w with the unused rth colour.

We now need to show that M_r cannot be coloured with $r - 1$ colours. Suppose we had such a colouring, say $g : V \to \{1, \ldots, r - 1\}$. We will construct a colouring g' of M_{r-1} with at most $r - 2$ colours, which will violate the inductive hypothesis. The vertex w has some colour c, and we know that none of the u_is have colour c. We define

$$g'(v_i) = \begin{cases} g(v_i) & g(v_i) \neq c \\ g(u_i) & g(v_i) = c \end{cases}.$$

That is, we recolour any v_i with colour c so that its colour matches that of u_i. We claim this gives a valid colouring of M_{r-1}, with only $r - 2$ colours since c is not used. Consider an edge $\{v_i, v_j\}$. We must show that $g'(v_i) \neq g'(v_j)$. Here are the possibilities:

(1) Neither $g(v_i)$ nor $g(v_j)$ is equal to c. In this case, neither one got changed, i.e. $g'(v_i) = g(v_i)$ and $g'(v_j) = g(v_j)$, and these two colours must be different as g was a valid colouring.
(2) Both $g(v_i)$ and $g(v_j)$ are equal to c. This is actually not possible, since g was assumed to be a valid colouring.
(3) Exactly one is equal to c, say $g(v_i) = c$ and $g(v_j) \neq c$. Then $g'(v_i) = g(u_i)$ and $g'(v_j) = g(v_j)$. Since g was assumed to be a valid colouring, and u_i is adjacent to v_j by construction, the two colours $g'(v_i)$ and $g'(v_j)$ are different.

Thus g' is a colouring of M_{r-1} with $r - 2$ colours, contradicting the inductive hypothesis. We conclude that M_r cannot be coloured with $r - 1$ colours, so $\chi(M_r) = r$. □

17. Not examinable: The Tree Enumeration Formula

This section was not covered in lecture. How many trees are there with n labeled vertices? (Equivalently, how many spanning trees does the complete graph K_n have?)

Example 17.1. On 2 vertices, just one. On 3 vertices, 3. On 4 vertices, there are $4!/2$ trees that are the path P_4, and 4 trees that are stars, for 16 total.

Doing 5 is not too bad either – there are $5!/2$ copies of P_4, 5 stars with 4 leaves, and $5 \cdot 4 \cdot 3$ 3-leaf stars where one leaf has sprouted another. The total is 125.

What is the sequence $1, 3, 16, 125$? They are all perfect powers, and you might guess n^{n-2}. (You hopefully verified the $n = 7$ case on Assignment 3!) This is correct. It is generally called Cayley's formula, though Cayley's paper points out that it was known earlier. There are many beautiful proofs of this formula, including a direct (but nonintuitive) bijective proof, and a proof involving linear algebra. I want to show you a nice proof by double-counting (i.e. calculating some appropriate quantity in two different ways) due to Pitman.

Theorem 17.2. *There are n^{n-2} trees on n labeled vertices.*

Proof. Recall that given a *rooted* tree, we can uniquely assign a direction to each edge so that all edges point away from the root.

Let X denote the set of *sequences* of directed edges that, when put together, make a rooted tree. We count $|X|$ in two ways.

First: We could find such a sequence by choosing a tree, then choosing its root (which tells us what the direction of each edge should be), then choosing in which order the edges appear in the sequence. The answer is $|X| = T_n \cdot n \cdot (n-1)!$, where T_n is the number of trees on n labeled vertices. This is because there are n ways to choose a root, and $(n-1)!$ ways to order the $n-1$ edges in the sequence.

Second, we could find such a sequence by starting with the empty graph and adding edges (with directions) one at a time. In the first step, there are $n(n-1)$ possibilities; n choices for the "out-vertex" with the arrow coming out of it, and $n-1$ choices for the "in-vertex" with the arrow coming into it (which can't be the out-vertex as this would create a loop).

For the second step, there are n choices for the out-vertex. For the in-vertex, it would seem to depend on the out-vertex. If the out-vertex is one of the two vertices from step 1, we had better avoid the other vertex from step 1 (which would create a cycle), leaving $n-2$ possibilities. Otherwise, we can choose any other vertex except the in-vertex from step 1 (which would create a vertex with two in-edges, impossible in a rooted tree). This is still $n-2$ choices though! We get $n(n-2)$.

This pattern continues, for the following reason. Suppose we have already drawn k edges. Then we have a forest of $n-k$ rooted trees. Now we want to add another direct edge by choosing an out-vertex and an in-vertex. There are n choices for out-vertex v. For the in-vertex, we need to *avoid* all vertices that are previous in-vertices (there are k of these, one from each step, and they are all distinct), *as well as* the vertex farthest upstream from v. (This could be v itself, and is well-defined as each vertex has at most 1 arrow pointing into it. We need to avoid this edge to avoid making a cycle.) Thus the number of possibilities is $n(n-(k+1))$.

Another way of saying this is that for the in-vertex, we need to choose the root of one of our $n-k$ rooted trees, but it cannot be the tree containing the out-vertex.

If we follow these rules, we really do get a rooted tree at the end. To see this, note that we have made sure not to add cycles, and we have added an in-edge to all but one of the vertices. Call that other one the root – we must have a connected graph because "swimming upstream" must eventually take you to the root.

We have shown $|X| = \prod_{k=1}^{n-1} n(n-k) = n^{n-1} \cdot (n-1)! = n^{n-2} \cdot n!$. Thus $T_n = n^{n-2}$. □

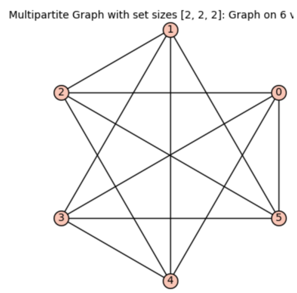
Remark 17.3. It is worth mentioning two things about the proof that uses linear algebra. First, it shows that Cayley's formula is a special case of a more general fact, as follows.

Let G be a simple graph with n vertices $v_1, \ldots, v_n$. We form an $n \times n$ matrix Q, called the *Laplacian* of G, by

$$Q_{ij} = \begin{cases} \deg(v_i) & i = j \\ -1 & i \neq j, \text{ and } i, j \text{ adjacent} \\ 0 & i \neq j, \text{ and } i, j \text{ not adjacent} \end{cases}.$$

Note that the sum of all rows (or columns) is zero. This means that Q has rank at most $n-1$. In fact the rank of Q is $n-c$, where c is the number of connected components of G — so if G is connected Q has rank $n-1$, and hence its eigenvalues are $0, \lambda_1, \ldots, \lambda_{n-1}$ for nonzero $\lambda_1, \ldots, \lambda_{n-1}$.

Example 17.4. The complete multipartite graph $K_{2,2,2}$ has the following Laplacian matrix:



$$\begin{pmatrix} 4 & 0 & -1 & -1 & -1 & -1 \\ 0 & 4 & -1 & -1 & -1 & -1 \\ -1 & -1 & 4 & 0 & -1 & -1 \\ -1 & -1 & 0 & 4 & -1 & -1 \\ -1 & -1 & -1 & -1 & 4 & 0 \\ -1 & -1 & -1 & -1 & 0 & 4 \end{pmatrix}$$

The characteristic polynomial is $x(x-4)^3(x-6)^2$, so the eigenvalues are $0, 4, 4, 4, 6, 6$.

Theorem 17.5 (Kirchoff). *G has exactly $\frac{1}{n}\lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_{n-1}$ spanning trees.*

Example 17.6. This says that $K_{2,2,2}$ has $\frac{1}{6}4^3 \cdot 6^2 = 384$ spanning trees.

Example 17.7. It is easy to check that the Laplacian matrix of K_n (which has $n-1$ on the diagonal, and -1 elsewhere) has characteristic polynomial $x \cdot (x-n)^{n-1}$. Thus K_n has $\frac{1}{n}n^{n-1} = n^{n-2}$ spanning trees.

Remark 17.8. The second thing to note about this proof is that it is actually related to circuit analysis. For example, Kirchoff used Theorem 17.5 to calculate the effective resistance between two points in a circuit of resistors. A special case says that if all resistors have the same resistance, the effective resistance between vertices a and b is proportional to the number of spanning trees of G/ab divided by the number of spanning trees of G, where G/ab is obtained by "gluing" a to b.

APPENDIX A

Support class problems

1. Week 2 support class problems

(1) You have 15 songs in your music library. A music service creates a 15-song playlist for you by choosing one song at random from your collection for each of the 15 slots. (So, for example, it is possible that it plays the same song 15 times, although that is exceedingly unlikely.)
   (a) How many possible playlists are there?
   (b) Of the playlists in part (a), what percentage of them repeat at least one song? First take a guess; then figure it out.
   (c) Of the playlists in part (a), what percentage of them repeat at least one song twice in a row? First take a guess; then figure it out.
   (d) Trickier, food for thought: What percentage of playlists play some song ≥ 3 times? What percentage of playlists play some song ≥ 3 times in a row? What if we replaced 3 by some other number, e.g. 4?

   (Note from M. Chan: On the whole, people tend to significantly underestimate the likelihood that a song repeats, or even repeats twice in a row. For this reason, music services like Spotify or iTunes initially fielded many complaints from users that their playlist algorithms were defective. In some cases, these companies decided to rewrite their algorithms to make their playlists feel more random, even though the new methods were actually less random!)

(2) Let n be a positive integer. Count the following objects:
   (a) Ordered pairs of subsets $A, B \subseteq \{1, \ldots, n\}$.
   (b) Nested ordered pairs of subsets $A \subseteq B \subseteq \{1, \ldots, n\}$.
   (c) Ordered pairs of subsets $A, B \subseteq \{1, \ldots, n\}$ such that $A \cap B \neq \emptyset$.
   (d) Ordered triples of subsets $A, B, C \subseteq \{1, \ldots, n\}$ such that $A \subseteq C$, $B \subseteq C$, and $A \cap B \neq \emptyset$.

(3) Prove that for every positive integer n, we have $\binom{2n}{n} < 4^n$. (Think combinatorially...)

Solutions.

(1) (a) $15^{15} \approx 4 \times 10^{17}$.
   (b) $\frac{15^{15}-15!}{15^{15}} = 99.9997\%$. (The number of playlists that do *not* repeat a song is 15!)
   (c) $\frac{15^{15}-15\cdot 14^{14}}{15^{15}} = 61.94\%$. (The number of playlists that do *not* repeat a song twice in a row is $15 \cdot 14^{14}$, since there are 15 choices for the first song, and 14 choices for all subsequent songs.)

(2) (a) There are 2^n subsets of $\{1, \ldots, n\}$, so there are $2^n \cdot 2^n = 4^n$ ordered pairs of subsets.

(b) For each element $i \in \{1, \ldots, n\}$, there are 3 possibilities: either $i \notin B$, or $i \in A$, or $i \in B \setminus A$. (And these choices are made independently.) Thus there are 3^n choices. To check this is okay: Call these three cases $P1, P2, P3$. Write down the bijection from length-n sequences of elements of $\{P1, P2, P3\}$ to nested pairs of subsets, and the bijection in the reverse direction. (Draw a Venn diagram! It will help with the remaining parts!)

(c) It is best to find the number of ordered pairs A, B with $A \cap B = \emptyset$, and subtract it from 4^n. In this case, each element is either in A, or in B, or in neither A nor B, giving 3^n choices. Thus the answer is $4^n - 3^n$.

(d) The total number of ordered triples (A, B, C) satisfying $A, B \subseteq C$ is 5^n, due to the five choices: "not in C", "in C but not A or B", "in A but not B", "in B but not A", and "in both A and B". The number of ordered triples (A, B, C) satisfying $A, B \subseteq C$ and $A \cap B = \emptyset$ is 4^n, due to the four choices: "not in C", "in C but not A or B", "in A", and "in B". Thus the number of ordered triples (A, B, C) satisfying $A, B \subseteq C$ and $A \cap B \neq \emptyset$ is $5^n - 4^n$. (You could write them all down when $n = 2$...)

(3) The left side is the number of n-element subsets of $\{1, \ldots, 2n\}$, and the right side is the number of *all* subsets.

2. Week 3 support class problems

(1) Let $F(n)$ be the number of set partitions of $[n]$ with no singleton blocks. Prove that $B(n) = F(n) + F(n+1)$.
(2) Find a simple closed form for the ordinary generating function of $a_n = n^2$.

Solutions.

(1) A set partition of $[n]$ either has no singleton blocks, or has at least one. We need to show that set partitions of $[n]$ with at least one singleton block are counted by $F(n+1)$. Given such a set partition, take all singleton blocks, combine them, and add the element $n+1$. This gives a set partition of $[n+1]$ with no singleton blocks. In the other direction, given a set partition of $[n+1]$ with no singleton blocks, take the block containing $n+1$, delete $n+1$, and split it into singletons. This gives a bijection between set partitions of $[n]$ with at least one singleton block and set partitions of $[n+1]$ with no singleton blocks, from which the expression follows. (There are other proofs.)
(2) One way: Begin with $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Taking a derivative gives

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}.$$

Multiply by x to get

$$\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} nx^n.$$

Repeat the two steps:

$$\frac{x(1+x)}{(1-x)^3} = \sum_{n=0}^{\infty} n^2x^n.$$

3. Week 4 support class problems

(1) Find closed formulas for $S(k,k-1)$, $S(k,2)$, and $S(k,k-2)$.
(2) Let $A(x)$ denote the ordinary generating function for a sequence $(a_0, a_1, a_2, \ldots)$. Find the generating functions for the following sequences:
   (a) $(a_0, 0, a_1, 0, a_2, 0, \ldots)$
   (b) $(a_0, a_0, a_1, a_1, a_2, a_2, \ldots)$
   (c) $(a_0, 0, a_2, 0, a_4, 0, a_6, 0, \ldots)$
   (d) $(0, a_1, 8 * a_2, 27 * a_3, 64 * a_4, 125 * a_5, \ldots)$
   (e) $(a_2 + 3 * a_1 + a_0, a_3 + 3 * a_2 + a_1, a_4 + 3 * a_3 + a_2, \ldots)$

Solutions.

(1) $S(k,k-1)$ counts set partitions of $[k]$ into $k-1$ parts. The sizes of the parts must be $1, 1, \ldots, 1, 2$. Such a set partitions is determined by which two elements of $[k]$ are in the 2-element part of the set partitions. Thus $S(k,k-1) = \binom{k}{2}$.

   $S(k,2)$ counts set partitions of $[k]$ into 2 parts. Such a set partition is determined by a proper nonempty subset of $[k]$, of which there are $2^k - 2$. But, each set partition gets counted exactly twice this way. (A set partition into two parts consists of two complementary sets S, T, and each contributes to the $2^{k-2} - 2$.) Thus $2S(k,2) = 2^k - 2$, i.e. $S(k,2) = 2^{k-1} - 1$.

   $S(k,k-2)$ counts set partitions of $[k]$ into $k-2$ parts. These either have sizes $1, 1, \ldots, 1, 2, 2$ or $1, 1, \ldots, 1, 3$. There are $\frac{1}{2}\binom{k}{2}\binom{k-2}{2} = \frac{k!}{2!2!(k-4)!} = \frac{k(k-1)(k-2)(k-3)}{4}$ of the first type, and $\binom{k}{3}$ of the second type, so we get $\frac{1}{2}\binom{k}{2}\binom{k-2}{2} + \binom{k}{3}$.

(2) (a) $A(x^2)$
   (b) $A(x^2) + xA(x^2)$
   (c) $\frac{A(x)+A(-x)}{2}$
   (d) $x\frac{d}{dx}\left(x\frac{d}{dx}\left(x\frac{d}{dx}\left(A(X)\right)\right)\right) = xA'(x) + 3x^2A''(x) + x^3A'''(x)$
   (e) $A(x) + \frac{3A(x)}{x} + \frac{A(x)}{x^2} - \frac{3a_0+a_1}{x} - \frac{a_0}{x^2}$.

4. Week 5 support class problems

(1) Let a_n denote the number of partitions of n into odd parts, where each part may appear no more than twice. Write $\sum_{n=0}^{\infty} a_n x^n$ as an infinite product. Use the product to calculate coefficients of powers of x up to x^7. You should get $a_7 = 3$, corresponding to 7, $5+1+1$, $3+3+1$.

Solution sketch.

(1) By similar reasoning to that we say in class, the product formula is

$$(1+x+x^2)(1+x^3+(x^3)^2)(1+x^5+(x^5)^2)\cdots = \prod_{i\geq 0}(1+x^{2i+1}+x^{4i+2}).$$

Expanding gives:

$$1+x+x^2+x^3+x^4+2x^5+2x^6+3x^7+3x^8+3x^9+4x^{10}+5x^{11}+6x^{12}+7x^{13}+\cdots$$

5. Weeks 6/7 support class problems

(1) Let $p_k(n)$ denote the number of integer partitions of n with exactly k parts, and let $p_k^{distinct}(m)$ denote the number of integer partitions of m into exactly k distinct parts.
  (a) List the integer partitions of 7 with 3 parts. List the integer partitions of 10 with 3 distinct parts. Conclude $p_3(7) = p_3^{distinct}(10)$.
  (b) Prove that
$$p_k(n) = p_k^{distinct}\left(n + \frac{k(k-1)}{2}\right)$$
  by defining an appropriate bijection. (Make sure you show that your proposed bijection is actually a bijection!)

(2) Let Q_n denote the hypercube graph, i.e. the graph whose vertices are length-n sequences of zeroes and ones, where two sequences are connected by an edge if they differ in only one place. How many edges does Q_n have? What is $\chi(Q_n)$?

Solution sketch.

(1) (a) We have $p_3(7) = 4$; the partitions are:
$$5+1+1, \quad 4+2+1, \quad 3+3+1, \quad 3+2+2.$$
  We also have $p_3^{distinct}(10) = 4$; the partitions are:
$$7+2+1, \quad 6+3+1, \quad 5+4+1, \quad 5+3+2.$$
  (b) We define a bijection F. Given a partition of n, call it $a_k + a_{k-1} + \cdots + a_2 + a_1$ with $a_k \geq a_{k-1} \geq \cdots \geq a_1$, we have F send this partition to the partition of $n + \frac{k(k-1)}{2}$ given by $(a_k + k - 1) + (a_{k-1} + k - 2) + \cdots + (a_2 + 1) + (a_1 + 0)$. That is, we add zero to the smallest part, one to the next smallest part, and so on.
  The reason we end with with a partition of $n + \frac{k(k-1)}{2}$ is that we have added $0 + 1 + 2 + \cdots + (k-1) = \frac{k(k-1)}{2}$ to our original partition of n.
  We claim that the parts of the output partition are distinct. Indeed, we know $a_i \geq a_{i-1}$, so $a_i + (i-1) > a_{i-1} + (i-2)$.
  Finally, F is a bijection because the above operation is reversible – given a partition of $n + \frac{k(k-1)}{2}$ into k distinct parts, we subtract $k-1$ from the largest part, $k-2$ from the next largest part, and so on. This is an inverse to the F.

(2) Given a length-n sequence of zeroes and ones, the number of ways to change a single entry is n. That is, Q_n is n-regular. Since Q_n has 2^n vertices, the degree-sum formula then says that Q_n has $\frac{n \cdot 2^n}{2} = n \cdot 2^{n-1}$ edges. We also have $\chi(Q_n) = 2$, since we can colour a sequence red if it has an even number of ones, and blue otherwise, and this gives a valid 2-colouring.

APPENDIX B

Graph Theory Summary

1. Definitions

Definition 1.1 (**Graph**). A (simple) graph is a pair $G = (V, E)$, where V is a finite set ("vertices") and E is a set ("edges") of unordered pairs $\{v, w\}$, where $v, w \in V$. (Sometimes we allow duplicate edges and loops, in which case G is called a *multigraph*.)

Definition 1.2 (**Degree**). The degree $\deg(v)$ of a vertex v of a graph is the number of edges[1] incident to v.

Definition 1.3 (**Regular graph**). A graph G is k-regular if $\deg(v) = k$ for every vertex v.

Definition 1.4 (**Subgraph**). A subgraph of a graph $G = (V, E)$ is a graph whose vertices are a subset of V and whose edges are a subset of E.

Definition 1.5 (**Spanning subgraph**). A subgraph H of $G = (V, E)$ is a spanning subgraph if H has vertex set V.

Definition 1.6 (**Induced subgraph**). Given a graph $G = (V, E)$ and a subset $U \subseteq V$, the subgraph of G induced by U is the subgraph of G consisting of the vertices U and *all* edges of G connecting vertices of U.

Definition 1.7 (**Isomorphic graphs, graph isomorphism, graph automorphism**). An isomorphism from a (simple) graph $G = (V, E)$ to a (simple) graph $G' = (V', E')$ is a bijection $\phi : V \to V'$ such that vertices $v_1, v_2 \in V$ are connected by an edge of G if and only if $\phi(v_1), \phi(v_2) \in V'$ are connected by an edge of G'. Two (simple) graphs are isomorphic if there exists an isomorphism between them. (That is, they are the same after renaming vertices.) An automorphism of a (simple) graph G is an isomorphism from G to itself. (Note: Isomorphisms of multigraphs are somewhat more complicated and we don't mention them.)

Definition 1.8 (**Walk, closed walk, path, cycle**). A walk in a graph is a sequence of the form

$$v_1, e_1, v_2, e_2, \ldots, v_{k-1}, e_{k-1}, v_k,$$

where e_i connects v_i to v_{i+1}. A closed walk is a walk such that $v_k = v_1$. (We often do not bother specifying the starting point of a closed walk.) A path in a graph is a walk with no repeated vertices[2]. A cycle in a graph is a closed walk with no other repeated vertices (other than the start/end)[3]. (Note: We haven't used the term *walk* much, and I won't use it on the exam without clarification.)

[1] In a multigraph, loops, if any, are counted twice

[2] Alternatively, a subgraph isomorphic to a path graph P_n

[3] Alternatively, a subgraph isomorphic to a cycle graph C_n. Note that 1-cycles and 2-cycles are impossible in a simple graph

Definition 1.9 (**Connected graph**). A graph G is connected if for any two vertices v_1, v_2, there is a path in G from v_1 to v_2. The maximal connected subgraphs of G are called connected components.

Definition 1.10 (**Tree, rooted tree, leaf**). A graph G is a tree if it connected and acyclic (contains no cycles). A rooted tree is a tree with a distinguished vertex. A leaf of a tree is a degree-1 vertex.

Definition 1.11 (**Colouring, chromatic number** $\chi(G)$). A (vertex) colouring of a graph G is an assignment of a colour to each vertex so that adjacent vertices have different colours. The chromatic number of a graph is the least number of colours used in any colouring.

Definition 1.12 (**Bipartite graph, multipartite graph**). A graph is k-partite if there exists a proper vertex colouring with k colours. (That is, G is k-partite if $\chi(G) \leq k$.)

Definition 1.13 (**Eulerian circuit, Eulerian graph**). An Eulerian circuit of a graph is a closed walk that traverses each edge exactly once. A graph that has an Eulerian circuit is caller Eulerian.

Definition 1.14 (**Hamiltonian cycle, Hamiltonian graph**). A Hamiltonian cycle of a graph is a closed walk that visits each vertex exactly once. A graph that has a Hamiltonian cycle is called Hamiltonian.

Definition 1.15 (**Independent set of vertices, vertex independence number** $\mathrm{ind}_V(G)$). A subset I of the vertices of a graph G is an independent set if no two elements of I are adjacent in G. The vertex independence number $\mathrm{ind}_V(G)$ of G is the largest possible size of an independent set of vertices.

Definition 1.16 (**Independent set of edges, matching, perfect matching, matching number** $\mathrm{ind}_E(G)$). A subset M of the edges of a graph G is an independent set, or a matching, if no two elements of M share an endpoint. A matching that spans G (touches every vertex) is a perfect matching. The edge independence number, or matching number, $\mathrm{ind}_E(G)$, of G is the largest possible size of a matching.

Definition 1.17 (**Planar**). A graph is planar if it can be drawn in $\mathbb{R}^2$ with no edge crossings.

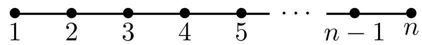
Definition 1.18 (**Edge contraction**). Given a graph G and an edge e, contracting e yields a new graph where e has been "shrunk to a point". That is, the edge e disappears, its two endpoints are merged, and all other connectivity remains as in G. (See Section 12 for several examples.)

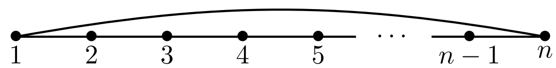
Definition 1.19 (**Minor**). A graph H is a minor of G if H can be obtained from G by a sequence of edge contractions, edge deletions, and vertex deletions. (Deleting a vertex also means deleting all incident edges.)

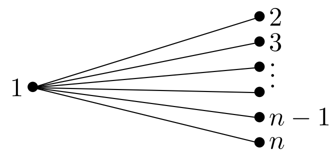
Definition 1.20 (**Ramsey number**). For positive integers k, ℓ, the Ramsey number $R(k, \ell)$ is the smallest n such that every 2-edge-colouring of K_n has either a red K_k or a blue K_ℓ.

2. Some examples of graphs:

EXAMPLE 2.1 (**Complete graph K_n, path P_n, cycle C_n, star S_n**).
Complete graph K_n: n vertices, any two distinct vertices connected by an edge.

Path P_n:


Cycle C_n:


Star S_n:


EXAMPLE 2.2 (**Complete bipartite $K_{a,b}$, complete multipartite $K_{a_1,\ldots,a_r}$**).
The complete bipartite graph $K_{a,b}$ has $a + b$ vertices

$$u_1, \ldots, u_a, v_1, \ldots, v_b,$$

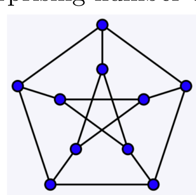
and an edge between u_i and v_j for all possible i and j.

The complete multipartite graph $K_{a_1,\ldots,a_r}$ has $a_1 + \cdots + a_r$ vertices

$$v_{1,1}, \ldots, v_{1,a_1}, v_{2,1}, \ldots, v_{2,a_2}, \ldots, v_{r,a_r},$$

and an edge between $v_{i,k}$ and $v_{i',k'}$ **if** $i \neq i'$, for all possible k and k'.

EXAMPLE 2.3 (**Petersen graph**). Here is the Petersen graph. It is often a useful example and has been extensively studied — historically, it has been the smallest counterexample to a surprising number of false conjectures.



3. Some problems involving graphs

These are the kinds of problems involving graph we have mentioned. For each one, I note some of our observations.

Graph colouring Problems.

PROBLEM 1 (Vertex colouring). Given a graph G, calculate the chromatic number $\chi(G)$.

- Easy lower bound: $\chi(G)$ is bounded below by the size of the largest complete subgraph in G.
- Easy upper bound: $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of a vertex. (Greedy algorithm exercise)
- Bipartite is the same, by definition, as $\chi(G) = 2$. This is equivalent to G having no odd cycles (Theorem 6.7).
- $\chi(K_n) = n$, and $\chi(C_n)$ is 2 (n even) or 3 (n odd).
- Four colour theorem (Theorem 13.1): If G is planar, $\chi(G) \leq 4$.

- $\chi(G)$ may be *much* larger than the size of the largest complete subgraph in G; The optional material Mycielski's Theorem (Theorem 16.1) gives graphs with high chromatic number with no triangles.

Graph Traversal Problems.

Problem 2 (Eulerian circuit). Given a graph G, decide whether G has an Eulerian circuit.

- Solved! Easy necessary and sufficient condition: Euler's Theorem (Theorem 8.4)

Problem 3 (Hamiltonian cycle). Given a graph G, decide whether G has a Hamiltonian cycle.

- Some easy necessary conditions, e.g. G must be connected and have no leaves.
- A sufficient condition (far from necessary): Dirac's Theorem (Theorem 9.5)

Graph "Packing Problems".

Problem 4 (Vertex Independence Number $\mathrm{ind}_V(G)$). Given a graph G, find the largest possible size of an independent set of vertices in G.

- $\mathrm{ind}_V(G)\chi(G) \geq |V|$, since each colour class is an independent set.

Problem 5 (Matching/Edge-Independence Number $\mathrm{ind}_E(G)$). Given a graph G, find the largest possible size of a matching in G.

- If there is a matching M with *no* unmatched vertices, then $\mathrm{ind}_E(G) = |M|$, and M called a perfect matching.
- Bipartite case: If A and B are the partite sets with $|A| \leq |B|$, best-case scenario is the existence of a matching of A. Necessary and sufficient condition: Hall's Theorem (Theorem 10.3).

Determining Planarity.

Problem 6. Given a graph G, is G planar?

- Necessary and sufficient condition: Kuratowski-Wagner Theorem (Theorem 12.6).

Ramsey Theory Problems.

Problem 7. What kinds of structures exist in all large graphs?

- Ramsey's Theorem (Theorem 14.5): A graph with sufficiently many vertices has a large complete subgraph or a large independent set.

Problem 8. What are the values of the Ramsey numbers $R(k, \ell)$?

- We showed $R(3, 3) = 6$ and $R(3, 4) \leq 10$ in lecture. ($R(3, 4) = 9$ is proved in the lecture notes.)
- The proof of Ramsey's Theorem (see Corollary 14.10) implies $R(k, \ell) \leq \binom{k+\ell-2}{k-1}$.
- Erdős gave (Theorem 14.13, a simple probabilistic argument) the lower bound $R(k, k) > 2^{k/2}$.

4. Main Theorems:

Proposition 2.5 (Degree-sum formula)**.** Let $G = (V, E)$ be a graph. Then[4] $\sum_{v \in V} \deg(v) = 2\,|E|$.

Theorem 5.14 (Characterization of trees)**.** Let $G = (V, E)$ be a graph. The following are equivalent:

(1) G is a tree. (That is, G is connected and acyclic.)
(2) G is connected and $|V| = |E| - 1$.
(3) G is acyclic and $|V| = |E| - 1$.
(4) Any two vertices in G are connected by a unique path.
(5) G is connected, but deleting any edge of G yields a disconnected graph.
(6) G is acyclic, but adding an edge between any two vertices of G yields a graph with a cycle.

Unexaminable: **Theorem 17.2** (Cayley's Formula, known at least as early as 1860 (Borchardt))**.** Given a (labeled) vertex set $V = \{v_1, \ldots, v_n\}$, there are n^{n-2} different trees with vertex set V.

Theorem 8.4 (Euler's Theorem, 1736)**.** A graph is Eulerian if and only if it is connected and every vertex has even degree.

Theorem 9.5 (Dirac's Theorem, 1952)**.** If every vertex of a simple graph $G = (V, E)$ has degree at least $|V|/2$, then G is Hamiltonian.

Theorem 6.7 (Characterization of bipartite graphs)**.** A graph is bipartite if and only if it has no odd cycles.

Unexaminable: **Theorem 16.1** (Mycielski's Theorem, 1955 (similar: Tutte 1947, Zykov 1949))**.** For all $r \geq 2$, the Mycielski graph M_r is triangle-free and has chromatic number r.

Theorem 10.3 (Hall's Theorem, 1935)**.** Let G be a bipartite graph with partite sets A and B. Then G has a matching of A if and only if G satisfies *Hall's condition*; that is, for every $S \subseteq A$ we have $|N_G(S)| \geq |S|$, where

$$N_G(S) = \{b \in B : b \text{ is adjacent to } s \text{ for some } s \in S\}.$$

Theorem 11.8 (Euler's Formula (Euler/Cauchy/etc, 1750-1811))**.** Let G be a planar graph, drawn with v vertices, e edges, and f faces. Then $v - e + f = 2$.

Corollary 11.11. A simple planar graph with n vertices has at most $3n - 6$ edges.

Corollary 11.13. A simple planar graph has a vertex with degree at most 5.

Corollaries 11.14 and 11.15. K_5 and $K_{3,3}$ are not planar.

Theorem 12.6 (Kuratowski-Wagner, 1930/1937)**.** A graph G is planar if and only if it does not have $K_{3,3}$ or K_5 as a minor.

Theorem 13.1 (Four/Five colour Theorem, Appel-Haken 1976-1989, Heawood 1890)**.** If G is a simple planar graph, then $\chi(G) \leq 4$. (Proved $\chi(G) \leq 5$ in lecture.)

Theorem 14.5 (Ramsey's Theorem, 1930)**.** Fix positive integers k and ℓ. Then for sufficiently large n, every 2-edge-colouring of K_n contains a red K_k or a blue K_ℓ.

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[4] In the case of non-simple graphs, make sure you count degrees correctly here!

Corollary 14.10 (Upper bound for Ramsey numbers)**.** The Ramsey number $R(k, \ell)$ satisfies $R(k, \ell) \leq \binom{k+\ell-2}{k-1}$.

Theorem 14.13 (Erdös lower bound for Ramsey numbers, 1947)**.** The Ramsey number $R(k, k)$ satisfies $R(k, k) > 2^{k/2}$.

Unexaminable: **Theorem 15.1** (Erdös 1959)**.** Fix positive integers k and r. Then there exists a graph with no cycles of length $\leq k$ whose chromatic number is at least r.