MA222 Metric Spaces
Lecture Notes 2023/2024

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1 Introduction

This module substantially generalizes the analysis on the real line numbers studied in previous analysys modules (Analysis I-III, Mathematical Analysis I-III). We will use Analysis to refer to material from any of these modules. You are expected to know this material. In addition, you are expected to know basic things about vector spaces.

1.1 Recommended Books

In addition to these notes, the following books cover the content of the course and are recommended.

- W. A. Sutherland, Introduction to Metric and Topological Spaces, Oxford University Press, first edition 1975, second edition 2009. (This is the top recommendation.)
- E. T. Copson, Metric Spaces, Cambridge University Press, first edition 1968
- W. Rudin, Principles of Mathematical Analysis, McGraw Hill, first edition 1953. (This is known in the trade as "baby Rudin", to distinguish from his more advanced text "Real and Complex Analysis". It is something of a classic but a Mathematical Association of America review in 2007 wrote "There is probably no more well known, respected, loved, hated, and feared text in all of mathematical academia". The review also contains the quotes "It was easy to say, and often true, that anyone who could survive a year of Rudin was a real mathematician" (Steven Krantz) and "Bourbakian propaganda, stripping and sterilizing analysis of any soul or meaning beyond the symbols" (Vladimir Arnold).)
- G. W. Simmons, Introduction to Topology and Modern Analysis, McGraw Hill. (More advanced, although it starts at the beginning; helpful for several third year and fourth year modules in analysis.)
- A. M. Gleason, Fundamentals of Abstract Analysis, first edition 1966 (Addison–Wesley), now published by CRC Press. (Only Chapter 14 is relevant to the module.)

1.2 Notation

We will use the following notation throughout the course.

- $\varnothing$ denotes the empty set.
- $\in$ denotes "is an element of".
- $\cup$ and $\cap$ denote union and intersection, respectively. (We will also use $\bigcup$ and $\bigcap$ for unions and intersections over families of sets.)
- $\subset$ denotes "is a subset of" (i.e. $A \subset B$ means that if $a \in A$ then $a \in B$). If A is a proper subset of B then we will write $A \subset B$ and $A \neq B$. (In practice, when the properness of a subset is important the fact that it is a subset is obvious so we'll only need to write $A \neq B$ or $B \setminus A \neq \varnothing$.)

- $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{R}$ the set of real numbers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{Z}$ the set of integers, and $\mathbb{N}$ the set of natural numbers $\{1, 2, 3, \ldots\}$. We also write $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$.
- For $a, b \in \mathbb{R}$, we denote open, closed and half-open intervals by $(a, b) = \{x \in \mathbb{R} : a < x < b\}$, $[a, b] = \{x \in \mathbb{R} : a \leq z \leq b\}$, $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$, $[a, b) = \{x \in \mathbb{R} : a \leq z < b\}$.
- For sets A and B, $A \times B$ will denote their Cartesian product, i.e. $A \times B = \{(a, b) : a \in A, \, b \in B\}$. (One must distinguish between (a, b) denoting an ordered pair, as here, with (a, b) denoting an open interval, as above.) This can be generalised to any finite product of sets.
- For $n \geq 2$, $\mathbb{R}^n$ is the Cartesian product of $\mathbb{R}$ with itself n times.
- Let A_α, $\alpha \in Y$, be a collection of sets indexed by an arbitrary set Y. Here, the product $\prod_{\alpha \in Y} A_\alpha$ is interpreted as the set of all functions $f : \mathcal{A} \to \bigcup_{\alpha \in Y} A_\alpha$ with the property that $f(\alpha) \in A_\alpha$. (To understand why, see Appendix 9.2.)
- If $A_\alpha = X$, for all $\alpha \in Y$, then we can write the above product as X^Y and note that it is the set of all functions $f : Y \to X$.
- A set A is countable if it is in bijection with a subset of $\mathbb{N}$ or, equivalently, if there is an injection from A to $\mathbb{N}$. A countable set may be finite or infinite, in which case we call it countably infinite. The empty set is countable.

1.3 Brief overview

The aim of the module is to generalise some of the concepts you have seen in earlier analysis modules to more abstract settings. Given a set X, we want to understand what it means for a sequence (x_n) in X to converge to $x \in X$, what we mean by taking a limit $\lim_{x \to a}$ in X and, given another set Y, what it means for $f : X \to Y$ to be continuous. All these concepts involve ideas of closeness and, in fact, without any additional structure on X and Y to measure closeness, they cannot be given any meaning. So it is fundamental to the subject that we will need to introduce some king of additional structure that measures closeness.

Let us think about $X = \mathbb{R}$ and convergence of sequences. Without getting into the ϵs, we know that "(x_n) converges to x" has the same meaning as "$|x_n - x|$ converges to zero" or, put more loosely, "the distance between x_n and x tends to zero". We could do the same think on any set provided we have a sensible notion of distance. The object we allow as distances are called *metrics* and are defined in section 3. They can be quite abstract and don't necessarily match our physical notion of distance. Therefore, before we get to the abstractions of metrics, the next section will discuss something which appears more natural: the notion of *norm* of vectors. Norms can be used to define distances and these turn out to be a special and natural class of metrics.

However, our abstract does not stop with metrics. It turns out that ideas of convergence and continuity can be expressed without using a distance, though there must be some substitute that allows us to think about closeness. This concept is called a *topology* and is introduced in section 5.

The later parts of the module, in sections 7, 8 and 9, introduce the three important concepts of compactness, connectedness and completeness but it is too early to describe what these things mean.

1.4 Sets, functions, images and pre-images.

In the course, we often have to look at the pre-images of sets under functions and it is worthwhile recapping the basic definitions and properties that we'll meet and use.

Let X and Y be sets and let $f : X \to Y$ be a function. The image $f(A)$ of a set $A \subset X$ is defined to be

$$f(A) = \{f(x) : x \in A\}.$$

Regardless of whether or not f is invertible, we define the pre-image $f^{-1}(B)$ of $B \subset Y$ to be

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

Our use of "f^{-1}" in the notation $f^{-1}(B)$ does not mean that f necessarily has an inverse. However, if f is invertible then

$$\{x \in X : f(x) \in B\} = \{f^{-1}(y) : y \in B\},$$

so $f^{-1}(B)$ is both the pre-image of B under f and the image of B under f^{-1}, and the notation is unambiguous.

We will use the following identities. If $f : X \to Y$ is a function and $\{B_i\}$ is an arbitrary collection of subsets of Y then

$$f^{-1}\left(\bigcup_i B_i\right) = \bigcup_i f^{-1}(B_i) \quad \text{and} \quad f^{-1}\left(\bigcap_i B_i\right) = \bigcap_i f^{-1}(B_i).$$

(Check that these follow from the definition of pre-image.) Furthermore, if f is invertible and $\{A_i\}$ is an arbitrary collection of subsets of X then

$$f\left(\bigcup_i A_i\right) = \bigcup_i f(A_i) \quad \text{and} \quad f\left(\bigcap_i A_i\right) = \bigcap_i f(A_i),$$

and the statement for the union still holds even if f is not invertible. However, the statement for the intersection need not hold if f is not invertible. For example, define $f : \mathbb{R} \to [0, \infty)$ by $f(x) = x^2$ and let $A_1 = (-\infty, 0]$, $A_2 = [0, -\infty)$. Then $f(A_1 \cap A_2) = f(\{0\}) = \{0\}$ but $f(A_1) \cap f(A_2) = [0, \infty) \cap [0, \infty) = [0, \infty)$.

1.5 Non-examinable content

Some parts of the lecture notes were not lectured or just not examinable in the year 2023-2024. Not examinable

These are highlighted like this.

Please note that occasionally a statement of a theorem is examinable,

but its proof is not. Not examinable

2 Normed Spaces

2.1 Norms

A *norm* on a vector space is a generalised notion of "length" of a vector. You will have seen examples before but let us make a general definition. Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1. A norm on a vector space X is a map $\|\cdot\| : X \to \mathbb{R}^+$ such that

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for every $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), $x \in X$ ("homogeneity"); and

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in X$ (the triangle inequality).

Note that if we started with only requiring $\|\cdot\| : X \to \mathbb{R}$ then in fact it follows automatically that $\|x\| \geq 0$ given (i)–(iii), since we have

$$0 = \|0\| = \|x + (-x)\| \leq \|x\| + \|-x\| = \|x\| + \|x\| = 2\|x\|$$

using first (i), then (iii), then (ii).

Example 2.2. We begin with what should be a familiar example (at least for $n = 2$ and 3). In the vector space $\mathbb{R}^n$, for $x = (x_1, \ldots, x_n)$ define

$$\|x\| = \left(\sum_{j=1}^{n} |x_j|^2 \right)^{1/2},$$

the "standard norm" or "Euclidean norm".

Let us check that this is indeed a norm. If $x = 0$ then $\|x\| = 0$, and if $\|x\| = 0$ then $|x_j| = 0$ for every j, i.e. $x = 0$, which gives (i). For (ii) we have

$$\|\lambda x\| = \left(\sum_{j=1}^{n} |\lambda x_j|^2 \right)^{1/2} = \left(\sum_{j=1}^{n} |\lambda|^2 |x_j|^2 \right)^{1/2} = |\lambda|\|x\|,$$

as required. It is usually the triangle inequality the requires some work to prove, and this is the case here.

Not examinable

Let $x \cdot y$ denote the usual "dot product": $x \cdot y = \sum_{j=1}^{n} x_j y_j$. We have

$$\begin{aligned} \|x + y\|^2 = (x + y) \cdot (x + y) &= \|x\|^2 + 2\, x \cdot y + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &\leq (\|x\| + \|y\|)^2 \end{aligned}$$

using the inequality $|x \cdot y| \leq \|x\|\,\|y\|$.

However, in addition to the Euclidean norm, there are many other possibilities for norms on $\mathbb{R}^n$. Here are two easy examples:

$$\|x\|_1 = \sum_{j=1}^{n} |x_j|$$

and

$$\|x\|_\infty = \max_{j=1,\ldots,n} |x_j|.$$

These are both norms on $\mathbb{R}^n$. Proving the triangle inequality for these norms is almost trivial, since $|x_j + y_j| \le |x_j| + |y_j|$. We will see more examples soon but first we develop some more theory.

Definition 2.3. If X is a vector space and $\|\cdot\|$ is a norm on X, the pair $(X, \|\cdot\|)$ is a *normed space.*

Many spaces have a "standard norm", so for example $\mathbb{R}^n$ is usually $\mathbb{R}^n$ with the Euclidean norm, or it is clear that we are working with a particular norm. Thus, we might talk about "the normed space X" rather than the normed space $(X, \|\cdot\|)$ since this is less of a mouthful.

The (closed) unit ball in $(X, \|\cdot\|)$ is the set

$$\overline{\mathfrak{B}}_X := \{x \in X : \ \|x\| \le 1\}.$$

(Note that this depends on which norm we choose but, for simplicity, we supress this in the notation. We are using a gothic letter $\mathfrak{B}$ here because we want to reserved the usual B for open balls, which appear later and which play a much more important role.)

We will now show that the unit ball is always convex (defined below). This will turn out to be a useful property.

Definition 2.4. Let X be a vector space. A subset K of X is *convex* if whenever $x, y \in K$ and $0 \le \lambda \le 1$ we have $\lambda x + (1-\lambda)y \in K$. (Put more informally, a set is convex if the line segment joining any two points in the set is entirely contained in the set.)

Lemma 2.5. *In any normed space $(X, \|\cdot\|)$, the closed unit ball $\overline{\mathfrak{B}}_X$ is convex.*

Proof. If $x, y \in \overline{\mathfrak{B}}_X$ then $\|x\| \le 1$ and $\|y\| \le 1$. Then, for $0 < \lambda < 1$,

$$\|\lambda x + (1-\lambda)y\| \le |\lambda|\|x\| + |1-\lambda|\|y\| \le \lambda + (1-\lambda) = 1,$$

so $\lambda x + (1-\lambda)y \in \overline{\mathfrak{B}}X$. □

[NB: In an entirely similar way the open unit ball $\{x \in X : \ \|x\| < 1\}$ is also convex.]

We now give a relatively simple way to check that a particular function defines a norm, based on the convexity of the "closed unit ball" that this function would give rise to.

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Lemma 2.6. *Suppose that a function $N : X \to \mathbb{R}^{\geq 0}$ satisfies* (i) *and* (ii) *of the definition of a norm and, in addition, that the set $\overline{\mathfrak{B}} := \{x \in X : N(x) \leq 1\}$ is convex. Then N satisfies the triangle inequality*

$$N(x+y) \leq N(x) + N(y)$$

and so defines a norm on X.

Proof. We only need to prove the triangle inequality. If $N(x) = 0$ then $x = 0$ and

$$N(x+y) = N(y) = N(x) + N(y),$$

so we can assume that $N(x) > 0$ and $N(y) > 0$.

In this case $x/N(x) \in \overline{\mathfrak{B}}$ and $y/N(y) \in \overline{\mathfrak{B}}$, so using the convexity of $\overline{\mathfrak{B}}$ we have

$$\frac{N(x)}{N(x)+N(y)}\left(\frac{x}{N(x)}\right) + \frac{N(y)}{N(x)+N(y)}\left(\frac{y}{N(y)}\right) \in \overline{\mathfrak{B}}.$$

So

$$\frac{x+y}{N(x)+N(y)} \in \overline{\mathfrak{B}},$$

which means, using property (ii) from Definition 2.1 that

$$N\left(\frac{x+y}{N(x)+N(y)}\right) = \frac{N(x+y)}{N(x)+N(y)} \leq 1 \quad \Rightarrow \quad N(x+y) \leq N(x) + N(y),$$

as required. □

To apply Lemma 2.6 in examples, we often have to use that some real-valued function is *convex*. Recall that a function $f : [a,b] \to \mathbb{R}$ is convex if whenever $x, y \in [a,b]$ we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \qquad \text{for all } 0 \leq t \leq 1. \tag{1}$$

If $f \in C^2((a,b)) \cap C^1([a,b])$ then a sufficient condition for convexity of f is that $f''(x) \geq 0$ for all $x \in (a,b)$ (see Problem Sheet 1). In particular, we will use the fact that the function $s \mapsto |s|^p$ is convex for all $1 \leq p < \infty$ (proof also on Problem Sheet 1).

We are now in a position to introduce more norms on $\mathbb{R}^n$.

Example 2.7. For $1 \leq p < \infty$ the p-norm on $\mathbb{R}^n$ is given by

$$\|x\|_p := \left(\sum_{j=1}^n |x_j|^p\right)^{1/p} \qquad 1 \leq p < \infty.$$

(The standard norm corresponds to the choice $p = 2$.) For $p = \infty$, we already defined the ∞-norm above:

$$\|x\|_\infty = \max_{j=1,\ldots,n} |x_j|.$$

We have $\|x\|_\infty = \lim_{p\to\infty} \|x\|_p$ for any $x \in \mathbb{R}^n$, so there is some consistency in these definitions (see Problems Sheet 1).

We need to check that $\|\cdot\|_{\ell^p}$ are really norms. Properties (i) and (ii) are easy to check (as we did for $\|\cdot\| = \|\cdot\|_{\ell^2}$ above) and the hard part is to show that the triangle inequality holds. In fact, this is a standard inequality, called Minkowski's inequality, and it is proved in the next lemma.

Lemma 2.8 (Minkowski's inequality in $\mathbb{R}^n$)**.** *For all $1 \le p \le \infty$, if $x, y \in \mathbb{R}^n$ then*

$$\|x+y\|_{\ell^p} \le \|x\|_{\ell^p} + \|y\|_{\ell^p}. \tag{2}$$

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Proof. If $p = \infty$ this is straightforward. For $1 \le p < \infty$ we will use Lemma 2.6 and show that the set

$$\mathfrak{B} := \{x \in \mathbb{R}^n : \ \|x\|_{\ell^p} \le 1\} = \{x \in \mathbb{R}^n : \ \|x\|_{\ell^p}^p \le 1\}$$

is convex. To do this, we use the fact that the function $t \mapsto |t|^p$ is convex (see Problem Sheet 1) for all $1 \le p < \infty$. If $x, y \in \mathfrak{B}$ then

$$\begin{aligned}\|\lambda x + (1-\lambda)y\|_{\ell^p}^p &= \sum_{j=1}^n |\lambda x_j + (1-\lambda)y_j|^p \\ &\le \sum_{j=1}^n \lambda |x_j|^p + (1-\lambda)|y_j|^p \le 1,\end{aligned}$$

and so $\lambda x + (1-\lambda)y \in \mathfrak{B}$ and $\mathfrak{B}$ is convex; inequality (2) now follows from Lemma 2.6. □

We note down a particular case of this which we will use later:

$$\{(\alpha_1+\beta_1)^p + (\alpha_2+\beta_2)^p\}^{1/p} \le (\alpha_1^p + \alpha_2^p)^{1/p} + (\beta_1^p + \beta_2^p)^{1/p}. \tag{3}$$

In some sense that we will return to later, all these norms on $\mathbb{R}^n$ are "very similar".

Definition 2.9. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are called *equivalent* if there exist constants $0 < c_1 \le c_2$ such that

$$c_1\|x\|_1 \le \|x\|_2 \le c_2\|x\|_1 \qquad \text{for every } x \in X.$$

It is easy to check that this is an equivalence relation.

Two norms are equivalent if and only if there exist constants $0 < c_1 \le c_2$ such that

$$c_1\mathfrak{B}_{(X,\|\cdot\|_2)} \subset \mathfrak{B}_{(X,\|\cdot\|_1)} \subset c_2\mathfrak{B}_{(X,\|\cdot\|_2)},$$

where $\mathfrak{B}_{(X,\|\cdot\|_j)}$ is the closed unit ball in $(X, \|\cdot\|_j)$, $j = 1, 2$, i.e. you can sandwich $\mathfrak{B}_{(X,\|\cdot\|_1)}$ between two scaled copies of $\mathfrak{B}_{(X,\|\cdot\|_2)}$.

The next family of examples are given by spaces of sequences.

Example 2.10. The sequence space ℓ^p, $1 \le p < \infty$, consists of all sequences $x = (x_j)_{j=1}^\infty$ such that

$$\sum_{j=1}^{\infty} |x_j|^p < \infty$$

("pth power summable sequences") equipped with the norm

$$\|x\|_{\ell^p} = \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p}; \tag{4}$$

and ℓ^∞ is the space of bounded sequences equipped with the norm

$$\|x\|_{\ell^\infty} = \sup_j |x_j|. \tag{5}$$

The spaces ℓ^p are all vector spaces, since if $x, y \in \ell^p$ we have

$$\begin{aligned} \sum_{j=1}^{n} |x_j + y_j|^p &\le \sum_{j=1}^{n} (2\max(|x_j|, |y_j|))^p \\ &\le \sum_{j=1}^{n} 2^p \max(|x_j|^p, |y_j|^p) \\ &\le 2^p \left(\sum_{j=1}^{n} |x_j|^p + \sum_{j=1}^{n} |y_j|^p \right), \end{aligned}$$

and so $\sum_{j=1}^\infty |x_j + y_j|^p < \infty$, i.e. $x + y \in \ell^p$. (The other vector space properties are trivial to check.)

These spaces are infinite dimensional: define for each $j \in \mathbb{N}$ the sequence

$$e^{(j)} = (0, 0, \dots, 1, 0, \dots)$$

which consists entirely of zeros apart from having 1 as its jth term. For any choice of $n \in \mathbb{N}$, the elements $\{e^{(1)}, \dots, e^{(n)}\}$ are linearly independent, which shows that ℓ^p is infinite dimensional.

Note that for any $1 \le q < p \le \infty$ there are elements of ℓ^p that are not elements of ℓ^q, e.g. the sequence $x = (x_j)_{j=1}^\infty$ with $x_j = j^{-1/q}$. So the spaces not only have different norms, they consist of different elements.

We can deduce that (4) and (5) do indeed define norms on ℓ^p using Lemma 2.8.

Lemma 2.11 (Minkowski's inequality in ℓ^p)**.** *For all $1 \le p \le \infty$ if $x, y \in \ell^p$ then $x + y \in \ell^p$ and*

$$\|x + y\|_{\ell^p} \le \|x\|_{\ell^p} + \|y\|_{\ell^p}. \tag{6}$$

Proof. The case $p = \infty$ is once again straightforward. For $p \in [1, \infty)$, given $x, y \in \ell^p$, we can use inequality (2) to guarantee that

Not examinable

$$\begin{aligned}\left(\sum_{j=1}^{n} |x_j + y_j|^p\right)^{1/p} &\leq \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{1/p} \\ &\leq \|x\|_{\ell^p} + \|y\|_{\ell^p};\end{aligned}$$

now we can take the limit as $n \to \infty$ to deduce (6). □

It is worth observing that these ℓ^p spaces are nested; the largest is ℓ^∞ and the smallest ℓ^1, see the Problems Sheet 1.

2.2 Subspaces

If $(X, \|\cdot\|)$ is a normed space and Y is a subspace of X, then $(Y, \|\cdot\|)$ is another normed space. (Strictly we define $\|\cdot\|_Y : Y \to [0, \infty)$ as the restriction of $\|\cdot\|$ to Y, i.e. $\|y\|_Y = \|y\|$ for every $y \in Y$.)

For example, c_0, the space of all null sequences, is a subspace of ℓ^∞. The space c_{00}, the space of all sequences with only a finite number of non-zero terms, is a subspace of ℓ^p for all $p \in [1, \infty]$. (See Problems Sheet 1.)

2.3 Spaces of continuous functions

We denote by $C([a, b])$ the space of (real-valued) continuous functions on the interval $[a, b]$. The usual norm to use on $C([a, b])$ is the supremum (maximum) norm

$$\|f\|_\infty := \sup_{x \in [a,b]} |f(x)| = \max_{x \in [a,b]} |f(x)|$$

(since any continuous function on a closed bounded interval attains its bounds).

Another family of norms are defined on $C([a, b])$ using an integral: for $p \in [1, \infty)$ set

$$\|f\|_{L^p} := \left(\int_a^b |f(x)|^p \, dx\right)^{1/p}.$$

(See Problems Sheet 1.)

3 Metric Spaces

In many situations we will be less concerned with the idea of length than with a generalised notion of "distance". We can also define a "distance" in a much more general setting.

3.1 Definition of a metric space and examples

Let X be any set. We will define a notion of distance between elements of X.

Definition 3.1. A *metric* d on a set X is a map $d : X \times X \to \mathbb{R}^+$ such that

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for every $x, y \in X$; and

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for every $x, y, z \in X$ (triangle inequality).

Note that given (i)–(iii) we could assume just that d maps into $\mathbb{R}$, since

$$0 = d(x, x) \leq d(x, y) + d(y, x) = d(x, y) + d(x, y) = 2d(x, y)$$

(using (i), then (iii), then (ii)).
We call (X, d) a *metric space*. (Often, we don't explicitly mention the metric and just write that X is a metric space.)

To relate this to the previous section, we can see that any norm $\| \cdot \|$ on a vector space X gives rise to a metric on X by setting $d(x, y) = \|x - y\|$.

Lemma 3.2. *If X is a vector space and $\| \cdot \| : X \to \mathbb{R}$ is a norm, then $d(x, y) := \|x - y\|$ is a metric on X.*

Proof. (i) If $x = y$ then $d(x, y) = \|x - y\| = 0$; if $d(x, y) = \|x - y\| = 0$ then $x = y$.
(ii) $d(x, y) = \|x - y\| = \|y - x\| = d(y, x)$.
(iii) $d(x, z) = \|x - z\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$. □

We will now give some examples.

Example 3.3. Take $X = \mathbb{R}^n$ with any one of the metrics

$$d_p(x, y) := \|x - y\|_{\ell^p}, \qquad 1 \leq p \leq \infty.$$

The "standard metric" or "Euclidean metric" on $\mathbb{R}^n$ is given by

$$d_2(x, y) = \|x - y\|_{\ell^2} = \left(\sum_{j=1}^{n} |x_j - y_j|^2 \right)^{1/2}.$$

This is the metric we use of $\mathbb{R}^n$ (or subsets of $\mathbb{R}^n$) if none is specified.

In the following examples, X is no longer required to be a vector space. You should check for yourself that they satisfy the definition of a metric.

Example 3.4. The *discrete metric* on any non-empty set X is defined by setting $d(x,x) = 0$ and $d(x,y) = 1$ if $x \neq y$. (This is useful for counterexamples, since it is very different from the metric arising from a norm.)

Example 3.5. Let X be the set of all genes (N character sequences of the four symbols ATGC). Then the *Hamming distance* between $x, y \in X$ is the number of different pairs. For example, for $N = 4$, $d(AAAC, AAAA) = 1$ and $d(TGAC, AGAA) = 2$. (You could define a similar metric on any collection of N-character sequences of n symbols, for any choice of $n \geq 2$.)

Example 3.6. Let X be the set of all words (finite sequences of $n = 26$ symbols). Then the Levenshtein (spelling) distance between x and y is the minimum number of 'edits' required to change from x to y, where an 'edit' is any one of (i) insertion of a symbol (ii) deletion of a symbol (iii) change of a letter.

Example 3.7. Let G be a graph (a set of vertices joined by edges). The combinatorial metric (also known as graph distance or shortest path distance) defined on the vertices of G is the minimal number of edges required to join the two vertices. (For this definition, we need to assume that each pair of vertices can be joined by a path in the graph.)

Example 3.8. Sunflower metric on $\mathbb{R}^2$:

$$d(x,y) = \begin{cases} \|x - y\| & \text{if } x \text{ and } y \text{ lie on same line through the origin} \\ \|x\| + \|y\| & \text{otherwise.} \end{cases}$$

Example 3.9. Jungle river metric on $\mathbb{R}^2$:

$$d((x_1,y_1),(x_2,y_2)) = \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2 \\ |y_1| + |x_1 - x_2| + |y_2| & \text{otherwise.} \end{cases}$$

3.2 Metrics on subsets and products

If (X,d) is a metric space and A is a subset of X, then $d|_A$ is also a metric on A, i.e. we can define $d_A : A \times A \to [0,\infty)$ by setting

$$d_A(a_1,a_2) := d(a_1,a_2), \qquad a_1, a_2 \in A.$$

In this case (A, d_A) is a (metric) subspace of (X,d), and is also a metric space in its own right: we usually just write (A,d) for simplicity.

Example 3.10. Consider the metric space $(\mathbb{R}, d_2)$. Then (A, d_2) is another metric space for any $A \subset \mathbb{R}$, e.g. $[0,1]$ (with the usual metric).

Question. *Consider $(\mathbb{R}^3, d_2)$, that is $\mathbb{R}^3$ in the standard (Euclidean) metric. Let $A \subset \mathbb{R}^3$ be a sphere of radius R. What is the distance of two antipodal (i.e. opposite, like North and South poles) points in (A, d_2)? Is it $2R$ or πR?*

Example 3.11. Let (X, d) be the set of all words (of arbitrary finite length) equipped with the spelling metric. Let X_N be the set of all words of fixed length N; then (X_N, d) is another metric space. It is interesting to note that d is not (in general) the same as the Hamming distance on X_N. For example, take $N = 4$ and consider the words HEAR and EARN. Then $d_{\text{Hamming}}(\text{HEAR}, \text{EARN}) = 4$ but, by performing the edits HEAR $\to$ EAR $\to$ EARN, we see that the spelling distance $d(\text{HEAR}, \text{EARN}) = 2$.

Given two sets X and Y, their product $X \times Y$ consists of all elements of the form (x, y), with $x \in X$ and $y \in Y$. If both X and Y have metrics, it is easy to define a metric on $X \times Y$; in fact we have quite a choice.

Lemma 3.12. *Let (X, d_X) and (Y, d_Y) be two metric spaces. Then, for any $1 \le p \le \infty$,*

$$\varrho_p\big((x,y),\,(x',y')\big) := \begin{cases} \left(d_X(x,x')^p + d_Y(y,y')^p\right)^{1/p} & \text{for } 1 \le p < \infty \\ \max\big(d_X(x,x'),\, d_Y(y,y')\big) & \text{for } p = \infty, \end{cases}$$

defines a metric on $X \times Y$.

The proof of the triangle inequality is a little painful to write down for a general p. It is very easy, though, if you take $p = 1$ when

$$\varrho_1\big((x,y),\,(x',y')\big) := d_X(x,x') + d_Y(x',y').$$

Proof. The is an exercise on Problem Sheet 2. □

One can also show that given any finite collection of metric spaces $\{(X_j, d_j) : j = 1, \dots, n\}$,

$$\varrho_p\big((x_1,\dots,x_n),\,(y_1,\dots,y_n)\big) = \left(\sum_{j=1}^{n} d_j(x_j,y_j)^p\right)^{1/p}$$

defines a metric on $\prod_{j=1}^{n} X_j = X_1 \times \cdots \times X_n$.

Question. *Consider $\mathbb{R}$ with the standard metric, that is, $d(x,y) = |x-y|$. Of course, $\mathbb{R} \times \mathbb{R}$ is just the two dimensional plane $\mathbb{R}^2$. For which choice of p does the metric ϱ_p defined above coincide with the standard (Euclidean) metric on $\mathbb{R}^2$?*

3.3 Open and closed sets

We now make some rather important definitions. Let (X, d) be an arbitrary metric space. The *open ball* centred at $a \in X$ of radius r is the set

$$B(a,r) = \{x \in X : \ d(x,a) < r\}$$

and the *closed ball* centred at $a \in X$ of radius r is the set

$$\overline{B}(a,r) := \{x \in X : \ d(x,a) \le r\}.$$

Here are some examples:

- The open ball of radius r centred at 0 in $\mathbb{R}$ is $(-r, r)$.
  The open ball of radius 1 centred at 0 in $[0, 2]$ is $[0, 1)$.
- If X is any set and d is the discrete metric then

$$B(x, r) = \begin{cases} \{x\} & \text{if } 0 < r \leq 1 \\ X & \text{if } r > 1. \end{cases}$$

- See Problem Sheet 1 for an example involving the sunflower metric.

Note that some care is required when thinking about open balls in relation to subspaces. For example, $B(0, 1)$ in $\mathbb{R}$ is $(-1, 1)$, but $B(0, 1)$ in the metric space $[0, 2]$ is $[0, 1)$.

Note that if $d(y, x) \leq r$ then

$$B(y, \varrho) \subset B(x, \varrho + r),$$

since $d(z, y) < \varrho$ implies that

$$d(z, x) \leq d(z, y) + d(y, x) < \varrho + r.$$

Definition 3.13. A subset S of (X, d) is *bounded* if there exist $a \in X$ and $r > 0$ such that $S \subset B(a, r)$. (The definition[1] remains unchanged if you insist that $a \in S$. See Problem Sheet 2.)

The next definition is very important.

Definition 3.14. A subset U of (X, d) is *open* (in X) if for every $x \in U$ there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. A subset F of (X, d) is *closed* (in X) if $X \setminus F$ is open.

Here are some easy examples:

- in $\mathbb{R}$ open intervals are open and closed intervals are closed (from Analysis);
- in any metric space (X, d) both X and $\varnothing$ are both open and closed;
- in a metric space with the discrete metric every point $\{x\}$ is open (take $\epsilon = 1/2$).

Question. *When is a set not open? Negate the definition to complete this sentence: U is not open if there exists a point $x \in U$ such that for every $\epsilon > 0$ ...*

We now look at some elementary properties of open sets.

Lemma 3.15. *Open balls are open.*

[1] The emptyset in any metric space is also defined to be bounded. So, more precisely, S in (X, d) is bounded if either $S = \varnothing$ or there is a point a in S (or X) such that $S \subset B(a, r)$ for some $r > 0$. Equivalently, a set S is bounded if there is a real number C such that $d(s, s') \leq C$ for every $s, s' \in S$.

Proof. Consider $B(a,r)$ for some $a \in X$, $r > 0$. Take $y \in B(a,r)$. Then $d(y,a) < r$; so $\epsilon := r - d(y,a) > 0$ and $B(y,\epsilon) \subset B(a,r)$, since if $d(z,y) < \epsilon$ we have

$$d(z,a) \leq d(z,y) + d(y,a) < \epsilon + d(y,a) = r.$$

□

Finite intersections of open sets are open.

Lemma 3.16. *If $U_1, \ldots, U_n$ are open in (X,d) then $\bigcap_{i=1}^n U_i$ is open in (X,d).*

Proof. Take $x \in \bigcap_{i=1}^n U_i$. Then for each $i = 1, \ldots, n$ we have $x \in U_i$, so there exists $\epsilon_i > 0$ such that $B(x,\epsilon_i) \subset U_i$. If we take $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$ then for every i

$$B(x,\epsilon) \subset B(x,\epsilon_i) \subset U_i,$$

so $B(x,\epsilon) \subset \bigcap_{i=1}^n U_i$. □

In contrast, the intersection of a countable number of open sets need not be open. For example, in $\mathbb{R}$ we have

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

which is not open.

Corollary 3.17. *If $F_1, \ldots, F_n$ are closed in (X,d) then $\bigcup_{i=1}^n F_i$ is closed in (X,d).*

Proof. Simply observe that

$$X \setminus \bigcup_{i=1}^n F_i = \bigcap_{i=1}^n (X \setminus F_i)$$

and apply Lemma 3.16 to see that $X \setminus \bigcup_{i=1}^n F_i$ is open. □

A countable union of closed sets need not be closed. For example, in $\mathbb{R}$

$$\bigcup_{n=1}^{\infty} \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right] = (-1,1),$$

which is not closed.

Unions of open sets and intersections of closed sets are better behaved. Any union of open sets is open.

Lemma 3.18. *If $\{U_i : i \in \mathcal{I}\}$ is any collection of sets that are open in (X,d), where $\mathcal{I}$ is any index set, then $U := \bigcup_{i \in \mathcal{I}} U_i$ is open in (X,d).*

Proof. If $x \in U$ then $x \in U_i$ for some $i \in \mathcal{I}$. Since U_i is open, there exists $\epsilon > 0$ such that $B(x,\epsilon) \subset U_i$, so $B(x,\epsilon) \subset U$ and U is open. □

Similarly, any intersection of closed sets is closed.

Corollary 3.19. *If $\{F_i : i \in \mathcal{I}\}$ is any collection of closed sets in (X,d) then $\bigcap_{i \in \mathcal{I}} F_i$ is closed in (X,d).*

Proof. Observe that

$$X \setminus \bigcap_{i \in \mathcal{I}} F_i = \bigcup_{i \in \mathcal{I}} (X \setminus F_i)$$

and apply Lemma 3.18 □

3.4 Convergence of sequences

We will now define convergence in a metric space and show that it can be understood in terms of open sets.

Definition 3.20. A sequence $(x_n)_{n=1}^{\infty}$ in (X,d) converges to $x \in X$ if

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

In terms of open balls, this can be phrased as for every $\epsilon > 0$ there exists $N \geq 1$ such that

$$x_n \in B(x,\epsilon) \qquad \text{for all } n \geq N.$$

Lemma 3.21. *A sequence in a metric space can have at most one limit.*

Proof. Suppose that, for a sequence $(x_k)_{k=1}^{\infty}$,

$$\lim_{k\to\infty} d(x_k, x) = \lim_{k\to\infty} d(x_k, y) = 0.$$

Then

$$0 \leq d(x,y) \leq d(x,x_k) + d(x_k,y) \to 0,$$

and so $d(x,y) = 0$, i.e. $x = y$. □

The above result may seem obvious but later we will see more general types of spaces where sequences can converge to more than one limit.

We now show that we can characterise convergence purely in terms of open sets, without involving the metric directly.

Lemma 3.22. *Let $(x_n)_{n=1}^{\infty}$ be a sequence in a metric space X. We have $x_n \to x$, as $n \to \infty$, if and only for every open set U containing x there is an $N \geq 1$ such that $x_n \in U$ for all $n \geq N$.*

Proof. If $x_k \to x$ and $U \ni x$ is open then $B(x,\epsilon) \subset U$ for some $\epsilon > 0$. There exists $N \geq 1$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$, i.e. such that $x_n \in B(x,\epsilon) \subset U$ for all $n \geq N$.

Conversely, suppose that for every open set U containing x there is an $N \geq 1$ such that $x_n \in U$ for all $n \geq N$. Then, given $\epsilon > 0$, the set $B(x,\epsilon)$ is an open set containing x, so there exists $N \geq 1$ such that $x_n \in B(x,\epsilon)$, for all $n \geq N$, i.e. such that $d(x_n, x) < \epsilon$, for all $n \geq N$. So we have shown $x_n \to x$. □

Lemma 3.23. *A subset F of a metric space is closed if and only if whenever a sequence $(x_n)_{n=1}^{\infty}$ contained in F converges to some $x \in X$, it follows that $x \in F$.*

Proof. Suppose that F is closed and that (x_n) is a sequence in F with $x_n \to x$. Assume, for a contradiction, that $x \notin F$. Since $X \setminus F$ is open, by Lemma 3.22 there exists $N \geq 1$ such that $x_n \in X \setminus F$ for all $n \geq N$; but this contradicts the fact that $x_n \in F$, and so we must have $x \in F$.

For the other direction, we prove the contrapostive. Suppose that F is not closed, i.e. $X \setminus F$ is not open. Then there exists some $x \in X \setminus F$ with the property that there is no $\epsilon > 0$ such that $B(x,\epsilon) \subset X \setminus F$. Then for each $k \in \mathbb{N}$ there exists $x_k \in B(x, 1/k)$ such that $x_k \notin X \setminus F$, i.e. such that $x_k \in F$. Then $x_k \to x$ but $x \notin F$. □

Exercise: Show that a closed ball $\{x \in X : d(a,x) \leq r\}$ is a closed set.

4 Continuity

4.1 Continuity in metric spaces

We begin by defining what it meant by the limit of a function between two metric spaces as the argument tends to a point. (N.B. We use the terms "function" and "map" interchangably.)

Definition 4.1. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be a function. For $p \in X$, we say that $\lim_{x \to p} f(x) = y \in Y$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$0 < d_X(x, p) < \delta \qquad \Rightarrow \qquad d_Y(f(x), y) < \epsilon.$$

Next, we define what it means for a function between two metric spaces to be continuous.

Definition 4.2. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be a function. Then f is

- *continuous at* $p \in X$ if $\lim_{x \to p} f(x) = f(p)$, i.e. if for every $\epsilon > 0$ there exists a $\delta > 0$ such that
$$d_X(x, p) < \delta \qquad \Rightarrow \qquad d_Y(f(x), f(p)) < \epsilon;$$
- *continuous* (on X) if it is continuous at every point of X.

Not examinable

There is a stronger notion called Lipschitz continuity.

Definition 4.3. A function $f : X \to Y$ is *Lipschitz continuous* or just *Lipschitz* if there exists $C \geq 0$ such that

$$d_Y(f(x), f(y)) \leq C d_X(x, y) \qquad \text{for ever} \quad x, y \in X.$$

We say that C is a *Lipschitz constant* (for f). It is easy to see that a Lipschitz continuous function is continuous (by taking $\delta = \epsilon / C$).

The following is a useful example of a Lipschitz function. Suppose $A \subset X$ is non-empty. Then we can define the distance of x from A by setting

$$d(x, A) = \inf_{z \in A} d(x, z).$$

Lemma 4.4. *Let (X, d) be a metric space. If $A \subset X$ is non-empty then the function $x \mapsto d(x, A)$ is Lipschitz with Lipschitz constant* 1.

Proof. Take $x, y \in X$. Then for every $z \in A$ we have

$$d(x, A) \leq d(x, z) \leq d(x, y) + d(y, z).$$

Taking the infimum over the right-hand side we obtain

$$d(x, A) \leq d(x, y) + d(y, A)$$

or

$$d(x, A) - d(y, A) \leq d(x, y).$$

We could run the same argument starting with $d(y, A) \leq d(y, x) + d(x, z)$, and so

$$|d(x, A) - d(y, A)| \leq d(x, y),$$

as required. □

Some basic properties of continuity that you have seen in Analysis continue to hold.

Lemma 4.5. *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X such that $x_n \to x \in X$ in (X, d_X), as $n \to \infty$. If $f : X \to Y$ is continuous at x then $f(x_n) \to f(x)$, as $n \to \infty$ in (Y, d_Y).*

Proof. Exercise. □

Lemma 4.6. *Let (X, d) be a metric space.*

1. *If $f, g : X \to \mathbb{R}$ are continuous then $f + g$ and fg are continuous and f/g is continuous at all points x where $g(x) \neq 0$.*
2. *If $(Y, \|\cdot\|)$ is a normed vector space and $f, g : X \to Y$ are continuous then $f + g$ is continuous.*

Proof. Exercise. (Note that we need Y to be a vector space in (2) so that $f + g$ is defined. If Y is only a metric space then $f + g$ need not be defined.) □

Note that we didn't specify a metric on Y. When Y is a normed space, you are expected to understand that the metric $d(y, y') = \|y - y'\|$ determined by the norm is being used (unless stated otherwise).

There is a close relationship between continuity and open sets but, before we explore this, we give some examples to show that the image of an open set under a continuous map need not be open.

- Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin x$. Then $f(-10, 10) = [-1, 1]$. Here, the image of an open set is closed.
- Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1/(1 + x^2)$. Then $f(\mathbb{R}) = (0, 1]$. Here, the image of an open set is neither open nor closed.

However, the *preimage* of any open set is open if f is continuous. If $f : X \to Y$ and $A \subset Y$ we write

$$f^{-1}(A) = \{x \in X : \ f(x) \in A\}$$

and call this the preimage of A (under f). The preimage of a set is defined even if f is not invertible. For example, take $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Then $f^{-1}(4, 9) =$

$(-3,-2) \cup (2,3)$. Furthermore, we can take the preimage of a set A even when A contains points that are not in $f(X)$: for the function $f(x) = 1/(1+x^2)$ considered above we have

$$f^{-1}((2,3)) = \varnothing$$

which is open; and

$$f^{-1}((1/2,2)) = f^{-1}((1/2,1]) = (-1,1)$$

which is open once again.

If f is invertible then $f^{-1} : f(X) \to X$ is defined and

$$f^{-1}(A) = \{f^{-1}(x) : \ x \in A \cap f(X)\}.$$

We now characterise continuity in terms of open sets.

Theorem 4.7. *Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous if and only if for any open set $U \subset Y$, $f^{-1}(U)$ is open in X.*

Proof. Suppose that f is continuous. Take any open set $U \subset Y$, and some point $x \in f^{-1}(U)$. Then $f(x) \in U$, which is open, so there exists $\epsilon > 0$ such that $B_Y(f(x), \epsilon) \in U$, i.e. $d_Y(f(x), y) < \epsilon$ implies that $y \in U$. Since f is continuous, there exists $\delta > 0$ such that $d_X(x', x) < \delta$ implies that $d_Y(f(x'), f(x)) < \epsilon$. So if $x' \in B_X(x, \delta)$ we have $f(x') \in B_Y(f(x), \epsilon) \subset U$, i.e. $B_X(x, \delta) \in f^{-1}(U)$. Hence $f^{-1}(U)$ is open.

Now, for the converse, suppose that $U \subset Y$ open implies $f^{-1}(U)$ open. Take $x \in X$ and $\epsilon > 0$. Then $B_Y(f(x), \epsilon)$ is open in Y, so $f^{-1}(B_Y(f(x), \epsilon))$ is open in X. Since this set contains x, we have $B_X(x, \delta) \subset f^{-1}(B_Y(f(x)), \epsilon))$ for some $\delta > 0$: but this inclusion says precisely that

$$d_X(x', x) < \delta \qquad \Rightarrow \qquad d_Y(f(x), f(x')) < \epsilon,$$

so that f is continuous at x. Since $x \in X$ is arbitrary, f is continuous. □

Now note that if $f : X \to Y$ then for any $A \subset Y$ we have

$$f^{-1}(Y \setminus A) = X \setminus f^{-1}(A).$$

(Check: If $x \in X$ such that $f(x) \in Y \setminus A$, then $x \notin f^{-1}(A)$, i.e. LHS $\subset$ RHS. If $x \notin f^{-1}(A)$ then $f(x) \notin A$, i.e. $f(x) \in Y \setminus A$ or $x \in f^{-1}(Y \setminus A)$, so RHS $\subset$ LHS.)

Using this we have the following corollary, defining continuity in terms of preimages of closed sets.

Corollary 4.8. *Theorem 4.7 holds with "open" replaced by "closed". In other words, a function $f : X \to Y$ is continuous if and only if for any closed set $F \subset Y$, $f^{-1}(F)$ is closed in X.*

Proof. This is an exercise on Problem Sheet 2. □

Another nice property is that continuity is closed under composition.

Lemma 4.9. *Suppose that (X, d_X), (Y, d_Y) and (Z, d_Z) are metric spaces and $f : X \to Y$ and $g : Y \to Z$ are continuous functions. Then $g \circ f : X \to Z$ is continuous.*

This result gives us a great illustration of how useful Theorem 4.7 is. First we will give a proof using the ϵ–δ definition of continuity; then we shall give a proof using the characterisation of continuity in terms of open sets. Observe how much easier the second proof is!

Not examinable

First proof. Take $a \in X$ and $\epsilon > 0$. Since g is continuous at $f(a)$ there exists $\delta_1 > 0$ such that

$$d_Y(y, f(a)) < \delta_1 \qquad \Rightarrow \qquad d_Z(g(y), g \circ f(a)) < \epsilon.$$

Since f is continuous at a, there exists $\delta_2 > 0$ such that

$$d_X(x, a) < \delta_2 \qquad \Rightarrow \qquad d_Y(f(x), f(a)) < \delta_1$$

so

$$d_X(x, a) < \delta_2 \qquad \Rightarrow \qquad d_Z(g \circ f(x), g \circ f(a)) < \epsilon.$$

□

Second proof. If U is an open subset of Z then $g^{-1}(U)$ is open in Y. So $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is open in X. □

4.2 Topologically equivalent metrics

Suppose we have two metrics d_1 and d_2 on the same set X. We have a definition of continuity that depends only on open sets, so if the open sets in (X, d_1) are the same as the open sets in (X, d_2), any function $f : X \to Y$ that is continuous using the d_1 metric on X should be continuous using the d_2 metric on X.

To prove this more formally, we can use Lemma 4.9 and the following observation: if d_1 and d_2 are two metrics on X, then the identity map $1_X : X \to X$ defined by $1_X(x) = x$ is continuous from (X, d_1) into (X, d_2) if and only if every set that is open in (X, d_2) is open in (X, d_1).

Lemma 4.10. *Suppose that d_1 and d_2 are two metrics on X. Then the following statements are equivalent:*

(i) every set that is open in (X, d_2) is open in (X, d_1);

(ii) for any metric space (Y, d_Y), if $g : X \to Y$ is continuous from (X, d_2) into (Y, d_Y) then g is continuous from (X, d_1) into (Y, d_Y); and

(iii) for any metric space (Y, d_Y), if $f : Y \to X$ is continuous from (Y, d_Y) into (X, d_1) then f is continuous from (Y, d_Y) into (X, d_2).

Proof. We show that (i) ⇒ (ii) and that (ii) ⇒ (i). The proof of (i) ⇔ (iii) is similar.

(i) ⇒ (ii): It follows from (i) that the identity map $1_X : (X, d_1) \to (X, d_2)$ is continuous. If $g : (X, d_2) \to Y$ is continuous then, by Lemma 4.9, $g \circ 1_X : (X, d_1) \to Y$ is continuous, and $g(1_X(x)) = g(x)$.

(ii) ⇒ (i): Take $(Y, d_Y) = (X, d_2)$ and $g = 1_X : (X, d_2) \to (X, d_1)$, i.e. $g(x) = x$. Since g is continuous from (X, d_2) into (X, d_2) it is continuous from (X, d_1) into (X, d_2). Thus, for every open set U in (X, d_2), $g^{-1}(U) = U$ is open in (X, d_1). □

As a consequence we have the following theorem (replacing logical $\Rightarrow$s in the above lemma by $\Leftrightarrow$s).

Theorem 4.11. *Suppose that d_1 and d_2 are two metrics on X. Then the following statements are equivalent:*

(i) the open sets in (X, d_1) and (X, d_2) coincide;

(ii) for any metric space (Y, d_Y), a function $g : X \to Y$ is continuous from (X, d_1) into (Y, d_Y) if and only if g is continuous from (X, d_2) into (Y, d_Y);

(iii) for any metric space (Y, d_Y), a function $f : Y \to X$ is continuous from (Y, d_Y) into (X, d_1) if and only if f is continuous from (Y, d_Y) into (X, d_2).

Definition 4.12. (i) Two metrics d_1 and d_2 on X are called *topologically equivalent*, or just *equivalent*, if the open sets in (X, d_1) and (X, d_2) coincide.
(ii) Two metrics d_1 and d_2 on X are called *Lipschitz equivalent* if there exist $0 < c \leq C < \infty$ such that

$$cd_1(x, y) \leq d_2(x, y) \leq Cd_1(x, y) \qquad \text{for all } x, y \in X.$$

Lemma 4.13. *Let d_1 and d_2 be two metrics on X that are* Lipschitz equivalent *on X. Then d_1 and d_2 are topologically equivalent.*

Proof. Exercise. □

Recall that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space X are said to be equivalent if there exist $0 < c \leq C < \infty$ such that

$$c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1.$$

Each norm induces a metric $d_i(x, y) = \|x - y\|_i$ on X. The following corollary is immediate from Lemma 4.13.

Corollary 4.14. *The metrics induced by equivalent norms are topologically equivalent.*

Example: The metrics induced by the ℓ^p norms on $\mathbb{R}^n$, $1 \leq p \leq \infty$, are topologically equivalent to each other (since the norms are equivalent).

Example: The metrics $d(x, y)$ and $d_1(x, y) := \min(d(x, y), 1)$ are topologically equivalent (they have the same open sets). (See Problem Sheet 2.)

As shown by the previous example, it is not true that topologically equivalent metrics are necessarily Lipschitz equivalent. However, for normed spaces we have the following.

Lemma 4.15. *If X is a vector space and two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X induce topologically equivalent metrics then the norms are equivalent.*

Not examinable

Proof. Since the metrics are topologically equivalent the identity map $1_X : (X, d_1) \to (X, d_2)$ is continuous; this is the same as considering the identity map between the two normed spaces $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$. In particular, the identity map is continuous at 0, so there exists $\delta > 0$ such that

$$\|x\|_1 < \delta \qquad \Rightarrow \qquad \|x\|_2 < 1.$$

For $y \in X$, take $x = \delta y/2\|y\|_1$, so that $\|x\|_1 = \delta/2 < \delta$. It follows that

$$\left\|\frac{\delta y}{2\|y\|_1}\right\|_2 < 1, \qquad \text{i.e.} \qquad \|y\|_2 < \frac{2}{\delta}\|y\|_1.$$

Likewise, we can use the fact that the identity map is continuous from $(X, \|\cdot\|_2)$ into $(X, \|\cdot\|_1)$ to show that $\|y\|_1 \leq (2/\delta')\|y\|_2$. □

Example: The norms $\|\cdot\|_{L^1}$ and $\|\cdot\|_{L^2}$ on $C[0, 1]$ are not topologically equivalent. (See Problem Sheet 2).

4.3 Isometries and homeomorphisms

Definition 4.16. Suppose that $f : X \to Y$ is a bijection such that

$$d_Y(f(x), f(y)) = d_X(x, y) \qquad \text{for all } x, y \in X.$$

Then f is called an *isometry* between X and Y. It preserves the distance between points, so X and Y are "the same" as metric spaces. We say that X and Y are *isometric*.

Definition 4.17. If $f : X \to Y$ is a bijection and both f and f^{-1} are continuous we say that f is a *homeomorphism* and that X and Y are *homeomorphic*.

If f is a homeomorphism then U is open in X if and only if $f(U)$ is open in Y. (Check: U open in Y implies that $f^{-1}(U)$ is open in X since f is continuous; V open in X implies that $f(V)$ is open in Y, since f^{-1} is continuous.)

Examples:

- X is homeomorphic to X (take $h = 1_X$).
- Any two open intervals (a, b) and (α, β) are homeomorphic; take

$$h(x) = \alpha + (\beta - \alpha)\frac{(x - a)}{(b - a)}.$$

- $(-1, 1)$ is homeomorphic to $\mathbb{R}$; take

$$h(x) = \tan(\pi x/2) \qquad \text{or} \qquad h(x) = x/(1 - |x|).$$

- Any open interval (a, b) is homeomorphic to $\mathbb{R}$; combine the two previous examples.

- The square is homeomorphic to the circle (construct a homeomorphism by "moving along rays").

Since the identity map $1_X : (X, d_1) \to (X, d_2)$ is always bijective two metrics on X are equivalent if and only if the identity map $1_X : (X, d_1) \mapsto (X, d_2)$ is a homeomorphism.

4.4 Topological properties I

If some property P of a metric space is such that if (X, d) has property P then so does every metric space that is homeomorphic to (X, d) we say that P is a *topological property*. More colloquially, these are properties that are only concerned with set-theoretic notions (e.g. countability) and/or open sets, rather than distances.

Examples of topological properties:

- X is open in X; X is closed in X;
- X is finite; countably infinite; or uncountable;
- X has a point such that $\{x\}$ is open in X (an 'isolated point');
- X has no isolated points;
- every subset of X is open;
- every continuous real-valued function on X is bounded.

Examples of properties that are not "topological":

- X is bounded;
- X is "totally bounded": for each $r > 0$ there exists a finite set F such that every ball of radius r contains a point of F.

5 Topological Spaces

5.1 Definition of a topology

We have seen that, in a metric space, convergence and continuity may be characterised purely in terms of the open sets. This prompts us to make the following definition, where we dispense with the need to have metric and only require that we have "open sets".

Definition 5.1. A *topology* $\mathcal{T}$ on a set T is a collection of subsets of T, which we agree to call the "open sets", such that

(T1) T and $\varnothing$ are open;

(T2) the intersection of finitely many open sets is open; and

(T3) arbitrary unions of open sets are open.

The pair $(T, \mathcal{T})$ is called a *topological space*.

Examples:

- The topology induced by a metric: in any metric space (X, d) the collection of all open sets forms a topology [using Lemmas 3.16 and 3.18].
- The *discrete topology*: all subsets are open (check: this is induced by the discrete metric).
- The *indiscrete topology*: the only open sets are T and $\varnothing$.
- The *co-finite topology*: a set is open if it is T, $\varnothing$, or its complement is finite.
- The *co-countable topology*: a set is open if it is T, $\varnothing$, or its complement is countable.
- The *Zariski topology* on $\mathbb{R}^n$: a set is open if it is $\mathbb{R}^n$, $\varnothing$, or its complement is the set of zeros of a polynomial with real coefficients.

Note that topologies need not come from metrics, i.e. there does not have to be a metric on T that gives rise to the same open sets. If there is, we say that $(T, \mathcal{T})$ is *metrisable* (we will return to this later). For now, we show that the indiscrete topology is not metrisable if T consists of more than one point.

Lemma 5.2. *Suppose that T consists of more than one point. Then the indiscrete topology on T is not metrisable.*

Proof. Suppose the indiscrete topology on T is induced by a metric d on T. Let $x, y \in T$ with $x \neq y$. Then $d(x, y) = \epsilon > 0$. The set $B(x, \epsilon/2)$ is an open subset of (T, d). Since $x \in B(x, \epsilon/2)$ this set is not empty, and since $y \notin B(x, \epsilon/2)$ this set is not all of T. But $\varnothing$ and T are the only open sets, so we have a contradiction. □

Sometimes (but not always) it is possible to compare two topologies on the same space.

Definition 5.3. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on T then we say that $\mathcal{T}_1$ is *coarser* than $\mathcal{T}_2$ if $\mathcal{T}_1 \subset \mathcal{T}_2$, i.e. $\mathcal{T}_1$ contains "fewer" open sets than $\mathcal{T}_2$. In this situation, we also say that $\mathcal{T}_2$ is *finer* than $\mathcal{T}_1$. [Coarser = smaller; finer = larger.] One can have two topologies on T that are not comparable (so they are not the same, but neither is finer than the other).

Sometimes we will say that a topology $\mathcal{T}$ is the smallest (or coarsest) topology with a given property. By this, we mean that if $\mathcal{T}'$ is another topology with this property then $\mathcal{T} \subset \mathcal{T}'$.

Closed sets in a topological space are simply defined to be the complements of open sets.

Definition 5.4. A subset of a topological space T is *closed* if its complement is open.

Using De Morgan's laws, the collection $\mathcal{F}$ of all closed sets satisfies

(F1) T and $\varnothing$ are closed;

(F2) the union of finitely many closed sets is closed; and

(F3) arbitrary intersections of closed sets are closed.

Given any collection $\mathcal{F}$ that satisfies these properties, we could take the collection of their complements as the open sets in some topology $\mathcal{T}$.

The co-finite topology (above) is more naturally specified in terms of its closed sets: T, $\varnothing$, and finite subsets. Properties (F1–3) are easily checked. Similarly for the co-countable topology and the Zariski topology.

5.2 Bases and sub-bases

In a metric space we do not have to specify all the open sets: we build them up from open balls (as seen on Problem Sheet 1). We can do something similar in a topological space.

Definition 5.5. A *basis* for a topology $\mathcal{T}$ on T is a collection $\mathcal{B} \subset \mathcal{T}$ such that every set in $\mathcal{T}$ is the union of some sets from $\mathcal{B}$, i.e. for all $U \in \mathcal{T}$, there exists $\mathcal{C}_U \subset \mathcal{B}$ such that $U = \bigcup_{B \in \mathcal{C}_U} B$.

Note that a collection of sets cannot be basis for two distinct topologies. (Suppose that $\mathcal{B}$ is a basis for both $\mathcal{T}$ and $\mathcal{T}'$. Then every set in $\mathcal{T}'$ is a union of sets in $\mathcal{B}$. Since $\mathcal{B} \subset \mathcal{T}$, this implies that $\mathcal{T}' \subset \mathcal{T}$. Similarly, $\mathcal{T} \subset \mathcal{T}'$.)

Example: Let us check that in a metric space open balls form a basis for the topology induced by the metric. A set U is open in a metric space (X, d) if and only if for every $x \in U$ there exists $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subset U$. So

$$U = \bigcup_{x \in U} B(x, \epsilon_x).$$

Thus, in a metric space, the collection of all open balls forms a basis for the topology induced by the metric.

The following lemma is an immediate consequence of the definition of a basis, since $T \in \mathcal{T}$, and if $B_1, B_2 \in \mathcal{B}$ then $B_1, B_2 \in \mathcal{T}$ so $B_1 \cap B_2 \in \mathcal{T}$.

Lemma 5.6. *If $\mathcal{B}$ is any basis for $\mathcal{T}$ then*

(B1) T is the union of some sets from $\mathcal{B}$ (i.e. there exists $\mathcal{C}_T \subset \mathcal{B}$ such that $\bigcup_{B \in \mathcal{C}_T} B = T$);

(B2) if $B_1, B_2 \in \mathcal{B}$ then $B_1 \cap B_2$ is the union of some sets from $\mathcal{B}$ (i.e. there exists $\mathcal{C}_{B_1 \cap B_2} \subset \mathcal{B}$ such that $\bigcup_{B \in \mathcal{C}_{B_1 \cap B_2}} B = B_1 \cap B_2$).

However, this can be reversed.

Proposition 5.7. *Let T be a set and let $\mathcal{B}$ be a collection of subsets of T that satisfy (B1) and (B2). Then there is a unique topology $\mathcal{T}$ on T whose basis is $\mathcal{B}$; its open sets are precisely the unions of sets from $\mathcal{B}$.*

Note that $\mathcal{T}$ is the smallest topology that contains $\mathcal{B}$.

Proof. If there is such a topology, then by the definition of a basis its sets consist of the unions of sets from $\mathcal{B}$. So we only need check that if $\mathcal{T}$ consists of unions of sets from $\mathcal{B}$ then this is indeed a topology on T. We check properties (T1–3).
(T1): T is the union of sets from $\mathcal{B}$ by (B1).
(T2): If $U, V \in \mathcal{T}$ then $U = \bigcup_{i \in \mathcal{I}} B_i$ and $V = \bigcup_{j \in \mathcal{J}} D_j$, with $B_i, D_j \in \mathcal{B}$. Then

$$U \cap V = \bigcup_{(i,j) \in \mathcal{I} \times \mathcal{J}} B_i \cap D_j,$$

which is a union of sets in $\mathcal{B}$ by (B2), and hence an element of $\mathcal{T}$.
(T3): Any union of unions of sets from $\mathcal{B}$ is a union of sets from $\mathcal{B}$. □

The following definition is also useful.

Definition 5.8. A *sub-basis* for a topology $\mathcal{T}$ on T is a collection $\mathcal{B} \subset \mathcal{T}$ such that every set in $\mathcal{T}$ is a union of finite intersections of sets from $\mathcal{B}$.

This is a bit more complicated to write explicitly than the definition of basis. It says that every $U \in \mathcal{T}$ can be written as a union $U = \bigcup_{i \in \mathcal{I}} D_i$, where each D_i has the form $D_i = B_1^{(i)} \cap \cdots \cap B_{n(i)}^{(i)}$, for some $B_1^{(i)}, \ldots, B_{n(i)}^{(i)} \in \mathcal{B}$.

(Note that if $\mathcal{B}$ is a basis for $\mathcal{T}$, then it is also a sub-basis for $\mathcal{T}$; but in general a sub-basis will be "smaller", i.e, have fewer sets.)

Example: The collection of intervals (a, ∞) and $(-\infty, b)$ (ranging over all $a, b \in \mathbb{R}$) is a sub-basis for the usual topology on $\mathbb{R}$, since intersections give the open intervals (a, b) and these are a basis for the usual topology.

Proposition 5.9. *If $\mathcal{B}$ is any collection of subsets of a set T whose union is T then there is a unique topology $\mathcal{T}$ on T with sub-basis $\mathcal{B}$. This $\mathcal{T}$ is precisely the collection of all unions of finite intersections of sets from $\mathcal{B}$.*

Proof. If $\mathcal{B}$ is a sub-basis for a topology $\mathcal{T}$, then this topology has $\mathcal{D}$, the collection of all finite intersections of elements of $\mathcal{B}$, as a basis. But $\mathcal{D}$ satisfies (B1) and (B2) from Lemma 5.6, so by Proposition 5.7 there is a unique topology $\mathcal{T}$ with basis $\mathcal{D}$, which is also the unique topology with sub-basis $\mathcal{B}$. □

Note that the topology $\mathcal{T}$ from this proposition is the smallest ('coarsest') topology on T that contains $\mathcal{B}$.

5.3 Subspaces and finite product spaces

Definition 5.10. If $(T, \mathcal{T})$ is a topological space and $S \subset T$, then the *subspace topology* on S is

$$\mathcal{T}_S = \{U \cap S : U \in \mathcal{T}\}.$$

We call $(S, \mathcal{T}_S)$ a (topological) subspace of T.

Example: If we consider $[0,1]$ as a subspace of $\mathbb{R}$ then the open sets in $[0,1]$ consist of all sets $U \cap [0,1]$ where U is an open subset of $\mathbb{R}$. In particular, $[0,a)$ is open for every $a \in (0,1)$ (that is, open in $[0,1]$ with the subspace topology).

We have already seen something similar when we restricted metrics to subspaces. The following lemma shows that this definition is consistent with our definition in metric spaces.

Lemma 5.11. *Suppose that (X, d) is a metric space with corresponding topology $\mathcal{T}$. If $S \subset X$ then the subspace topology $\mathcal{T}_S$ on S corresponds to the topology on S that arises from the metric space $(S, d|_S)$.*

Recall that d_S is the restriction of d to S, i.e. $d|_S(x,y) = d(x,y)$ for all $x, y \in S$. For simplicity we now write d_S for $d|_S$.

The key to the proof is that if $a \in S$ we have

$$\begin{aligned} B_X(a,\epsilon) \cap S &= \{x \in X : d(x,a) < \epsilon \text{ and } x \in S\} \\ &= \{x \in S : d(x,a) < \epsilon\} \\ &= \{x \in S : d_S(x,a) < \epsilon\} \\ &= B_S(a,\epsilon). \end{aligned}$$

Not examinable

Proof. Suppose that $a \in V \in \mathcal{T}_S$. Then $V = U \cap S$ for some $U \in \mathcal{T}$. Since U is open in (X, d), there exists $\epsilon > 0$ such that $B_X(a,\epsilon) \subset U$; so

$$B_X(a,\epsilon) \cap S \subset U \cap S.$$

But $B_X(a,\epsilon) \cap S = B_S(a,\epsilon)$, so

$$B_S(a,\epsilon) \subset U \cap S = V.$$

Hence, V is open in $(S, d|_S)$.

Conversely, suppose that V is open in (S, d_S). Then for any $a \in V$ we can find $\epsilon(a) > 0$ such that

$$B_S(a,\epsilon(a)) = B_X(a,\epsilon(a)) \cap S \subset V.$$

Now take the union over $a \in V$ to obtain

$$V = \bigcup_{a \in V} B_S(a,\epsilon(a)) = \left[\bigcup_{a \in V} B_X(a,\epsilon(a))\right] \cap S \in \mathcal{T}_S,$$

since the union in square brackets is an element of $\mathcal{T}$. □

Definition 5.12. Suppose that $(T_1, \mathcal{T}_1)$ and $(T_2, \mathcal{T}_2)$ are two topological spaces. Then the *product topology* on $T_1 \times T_2$ is the topology $\mathcal{T}$ with basis

$$\mathcal{B} = \{U_1 \times U_2 : \ U_1 \in \mathcal{T}_1, \ U_2 \in \mathcal{T}_2\}.$$

We call $(T_1 \times T_2, \mathcal{T})$ the (topological) product of T_1 and T_2.

(Note that the collection $\mathcal{B}$ can be a basis, since it satisfies (B1) and (B2) from Lemma 5.6. (B1) $T_1 \times T_2$ is a union of sets in $\mathcal{B}$ (in fact $T_1 \times T_2$ is an element of $\mathcal{B}$); and (B2) if we take $V, W \in \mathcal{B}$ with

$$V = V_1 \times V_2 \qquad W = W_1 \times W_2$$

then we have

$$V \cap W = (V_1 \cap W_1) \times (V_2 \cap W_2) \in \mathcal{B},$$

as required.)

Definition 5.12 extends to any finite number of factors.

Note that since $\mathcal{B}$ is a basis, $\mathcal{T}$ will in general contain far more sets than just those from $\mathcal{B}$.

The product topology is consistent with the definition of metrics on a product space we defined earlier.

Lemma 5.13. *If (X_1, d_1) and (X_2, d_2) are two metric spaces and $\mathcal{T}_1$ and $\mathcal{T}_2$ the corresponding topologies on X_1 and X_2, then for any choice of p with $1 \le p \le \infty$ the topology induced by the metric ϱ_p on $X_1 \times X_2$ defined in Lemma 3.12 coincides with the product topology $\mathcal{T}$ on $X_1 \times X_2$ as defined in Definition 5.12.*

Not examinable

Proof. For simplicity of notation we prove the result for the case

$$\varrho((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2),$$

i.e. we take $p = 1$. But the argument for other values of p is almost identical. (All the ϱ_p metrics are topologically equivalent.)

Suppose that $V \in \mathcal{T}$; we want to show that V is open when we use the ϱ-metric on $X_1 \times X_2$. Since $V \in \mathcal{T}$ we have

$$V = \bigcup_i U_1^i \times U_2^i,$$

with $U_1^i \in \mathcal{T}_1$ and $U_2^i \in \mathcal{T}_2$. So given any $(x_1, x_2) \in V$ such that $(x_1, x_2) \in U_1^i \times U_2^i$ for some i, there exists ϵ_1 such that $B_{d_1}(x_1, \epsilon_1) \subset U_1^i$ and ϵ_2 such that $B_{d_2}(x_2, \epsilon_2) \subset U_2^i$. It follows that

$$B_\varrho((x_1, x_2), \min(\epsilon_1, \epsilon_2)) \subset U_1^i \times U_2^i \subset V,$$

so V is open in the ϱ-metric.

Now we show that if $U \in X_1 \times X_2$ is open in the ϱ-metric then it is an element of $\mathcal{T}$, by showing that U can be written as

$$U = \bigcup_{(x_1,x_2)\in U} B_{d_1}(x_1, \epsilon(x_1,x_2)) \times B_{d_2}(x_2, \epsilon(x_1,x_2)) \tag{7}$$

for some $\epsilon(x_1,x_2) > 0$. This is sufficient, as all d_j-open balls are contained in $\mathcal{T}_j$, $j = 1, 2$. Given any $(x_1,x_2) \in U$, there exists $\epsilon > 0$ such that $B_\varrho((x_1,x_2),\epsilon) \subset U$. By the definition of ϱ, we have

$$B_{d_1}(x_1,\epsilon/2) \times B_{d_2}(x_2,\epsilon/2) \subset B_\varrho((x_1,x_2),\epsilon) \subset U;$$

taking the union over all $(x_1,x_2) \in U$ now yields equation (7). □

This allows us to give an example of the fact that elements of the product topology are not all of the form $U_1 \times U_2$ for $U_i \in \mathcal{T}_i$. Consider $B(0,1)$ in $\mathbb{R}^2$, which is open in the product topology (it is open for the usual metric on $\mathbb{R}^2$, which is the 'product metric' ϱ_p with $p = 2$). Suppose that $B(0,1) = U_1 \times U_2$. Then $(3/4, 0) \in B(0,1)$, so $3/4 \in U_1$, and $(0, 3/4) \in B(0,1)$, so $3/4 \in U_2$: but then $U_1 \times U_2 \ni (3/4, 3/4) \notin B(0,1)$.

5.4 Closure, interior and boundary

Let $(T, \mathcal{T})$ to be a topological space.

Definition 5.14. A *neighbourhood* of $x \in T$ is a set $H \subset T$ such that $x \in U \subset H$ for some $U \in \mathcal{T}$. An *open neighbourhood* of $x \in T$ is an open set U (i.e. a set $U \in \mathcal{T}$) that contains x.

Some authors use "neighbourhood" for "open neighbourhood".

Definition 5.15. The *closure* $\overline{A}$ of a set $A \subset T$ is the intersection of all closed sets that contain A.

Note that if A is non-empty then $\overline{A}$ is non-empty. Furthermore, $\overline{A}$ is always closed, since it is the intersection of closed sets. The closure of A is therefore the smallest closed set that contains A. (By "smallest", we mean that if F is a closed set that contains A then $\overline{A} \subset F$.) It follows that A is closed if and only if $A = \overline{A}$; so we have $\overline{\overline{A}} = \overline{A}$, since $\overline{A}$ is always closed.

Example: In $\mathbb{R}$, we have $\overline{(a,b)} = [a,b]$. We will soon be able to show easily that $\overline{\mathbb{Q}} = \mathbb{R}$ (see lemma below).

It follows almost immediately from the definition that

$$H \subset K \quad \Rightarrow \quad \overline{H} \subset \overline{K}$$

and

$$\overline{H \cup K} = \overline{H} \cup \overline{K}.$$

(See Problem Sheet 3.)

We can give the following alternative characterisations of closure.

Lemma 5.16. *The closure of A, $\overline{A}$, is the set*

$$\begin{aligned}\overline{A} &:= \{x \in T : \ U \cap A \neq \varnothing \text{ for every open set } U \text{ that contains } x\} \\ &= \{x \in T : \text{ every (open) neighbourhood of } x \text{ intersects } A\}.\end{aligned}$$

(We say that "A intersects B" if $A \cap B \neq \varnothing$.)

Proof. Let $x \in \overline{A}$. Suppose there is an open set U such that $x \in U$ and $U \cap A = \varnothing$, then $T \setminus U \supset A$. Since $T \setminus U$ is closed, we have $\overline{A} \subset T \setminus U$. However, this gives a contradiction, since $x \in \overline{A} \cap U$ and so $\overline{A} \cap U \neq \varnothing$. Therefore

$$\overline{A} \subset \{x \in T : \ U \cap A \neq \varnothing \text{ for every open set } U \text{ that contains } x\}.$$

Now suppose $x \in T$ is such that $U \cap A \neq \varnothing$ for every open set that contains x, but $x \notin \overline{A}$. Then $x \notin F$ for some closed set that contains A. So we have an open set $T \setminus F$ which contains x and satisfies $(T \setminus F) \cap A = \varnothing$, a contradiction. Therefore,

$$\{x \in T : \ U \cap A \neq \varnothing \text{ for every open set } U \text{ that contains } x\} \subset \overline{A}.$$

□

It follows easily from this lemma that in $\mathbb{R}$ we have $\overline{\mathbb{Q}} = \overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$. This shows that in general

$$\overline{H \cap K} \neq \overline{H} \cap \overline{K},$$

e.g. take $H = \mathbb{Q}$ and $K = \mathbb{R} \setminus \mathbb{Q}$; then $\overline{H} = \overline{K} = \mathbb{R}$, but $H \cap K = \varnothing$.

In a metric space we have a simple characterisation of the closure.

Lemma 5.17. *If X is a metric space and $A \subset X$ then*

$$\overline{A} = \{\text{limits of convergent sequences in } A\}.$$

Proof. If $(a_n)_{n=1}^{\infty}$ is a sequence in A then it is also a sequence in $\overline{A}$. If the sequence converges to $a \in X$ then, since $\overline{A}$ is closed and applying Lemma 3.23, we have $a \in \overline{A}$. On the other hand, if $a \in \overline{A}$ then, for every $n \geq 1$, we have $B(a, 1/n) \cap A \neq \varnothing$, so there exists $x_n \in A$ with $d(x_n, a) < 1/n$. Clearly $x_n \to a$, as $n \to \infty$. □

Definition 5.18. The *interior* of A, A°, is the union of all open subsets of A.

Since A° is the union of open sets it is open, and contained in A. It is the largest open subset of A. ("Largest" means that if $U \subset A$ is open then $U \subset A^\circ$.) So A is open if and only if $A = A^\circ$. It follows that $(A^\circ)^\circ = A^\circ$, since A° is open.

Again, it is more or less immediate from the definition that

$$H \subset K \quad \Rightarrow \quad H^\circ \subset K^\circ$$

and that

$$(H \cap K)^\circ = H^\circ \cap K^\circ.$$

(See Problem Sheet 3.)

Lemma 5.19. *The interior of A, A° consists of all points for which A is a neighbourhood, i.e.*

$$\{x \in T : \ x \in U \subset A \text{ for some } U \in \mathcal{T}\}.$$

Proof. If $x \in A^\circ$ then $x \in U$ for some open set $U \subset A$, i.e. A is a neighbourhood of x. Conversely, if A is a neighbourhood of x then there is an open set U such that $x \in U \subset A$, and so $x \in A^\circ$. □

We note in general

$$(H \cup K)^\circ \neq H^\circ \cup K^\circ :$$

we can use the same two sets as last time, $H = \mathbb{Q}$ and $K = \mathbb{R} \setminus \mathbb{Q}$. Then $H^\circ = K^\circ = \varnothing$, but $(H \cup K)^\circ = \mathbb{R}^\circ = \mathbb{R}$.

We can relate closures and interiors as follows.

Lemma 5.20. *If $A \subset T$ then*

$$A^\circ = T \setminus \overline{(T \setminus A)} \qquad \text{and} \qquad \overline{A} = T \setminus (T \setminus A)^\circ.$$

Proof. If $x \in A^\circ$ then A is a neighbourhood of x that does not intersect $T \setminus A$, so $x \notin \overline{T \setminus A}$, so $x \in T \setminus \overline{(T \setminus A)}$. If $x \in T \setminus \overline{(T \setminus A)}$ then $x \notin \overline{(T \setminus A)}$, so there is an open set containing x that does not meet $T \setminus A$. So this open set (which contains x) is a subset of A, so $x \in A^\circ$. Hence $A^\circ = T \setminus \overline{(T \setminus A)}$. The other part is left as an exercise on Problem Sheet 3. □

The next concept we want to introduce in this part is that of the boundary of a set.

Definition 5.21. The boundary ∂H of a set H is the set of all points x with the property that every neighbourhood of x meets both H and its complement:

$$\partial H = \{x \in T : \text{ if } U \text{ is an open set that contains } x \text{ then } U \cap H \neq \varnothing \text{ and } U \cap (T \setminus H) \neq \varnothing\}.$$

It is immediate from the definition that

$$\partial H = \overline{H} \cap \overline{T \setminus H},$$

so ∂H is always closed.

Using Lemma 5.20 we have $\overline{T \setminus H} = T \setminus H^\circ$, so we also have

$$\partial H = \overline{H} \cap (T \setminus H^\circ) = \overline{H} \setminus H^\circ.$$

Examples: In $\mathbb{R}$ we have $\partial(a, b) = \partial[a, b] = \{a, b\}$; $\partial\mathbb{Q} = \mathbb{R}$.

Not examinable

Definition 5.22. Let $S \subset T$. A point $x \in T$ is a *limit point* of S if every neighbourhood of x intersects $S \setminus \{x\}$. (Note that a limit point of S does not need to belong to S.) A point in S that is not a limit point of S is called an *isolated point*.
(If (X, d) is a metric space and $S \subset X$ then $x \in X$ is an *limit point* of S if, given $\epsilon > 0$, there exists $y \in S$ with $y \neq x$ such that $d(x, y) < \epsilon$. Equivalently, if for every $\epsilon > 0$, $(B(x, \epsilon) \setminus x) \cap S \neq \varnothing$.)

Examples:

- If $S = (0, 1) \subset \mathbb{R}$ then any point in the interval $[0, 1]$ is a limit point of S.
- If $S = [0, 1] \cup \{2\}$ then $\{2\}$ is not a limit point of S, so is an isolated point of S.

Note that if S is closed then it contains all its limit points, for otherwise we would have a limit point $x \in T \setminus S$, which would be an open set containing x that does not intersect $S = S \setminus \{x\}$.
We have $\overline{S} = S \cup \{\text{all limit points of } S\}$, see Problem Sheet 3.

Definition 5.23. A subset A of T is

- *dense* in T if $\overline{A} = T$;
- *nowhere dense* in T if $(\overline{A})^\circ = \varnothing$;
- *meagre in T* (or "of the first category in T") if it is a union of a countable number of nowhere dense sets. Not examinable

Examples: $\mathbb{Q}$ is dense in $\mathbb{R}$ (as is $\mathbb{R} \setminus \mathbb{Q}$). In $\mathbb{R}$, one-point sets are nowhere dense; so $\mathbb{Q}$ is meagre in $\mathbb{R}$. However, $\mathbb{R} \setminus \overline{\mathbb{Q}} = \varnothing$, so $\mathbb{Q}$ isn't nowhere dense.

Note: equivalently, a subset A of T is nowhere dense if $T \setminus \overline{A}$ is dense in T, since (using Lemma 5.20) we have

$$(\overline{A})^\circ = T \setminus \overline{(T \setminus \overline{A})}.$$

(If A is closed this is just $T \setminus A$ is dense.)

5.5 The Cantor set

We will now discuss the construction of an interesting subset of $\mathbb{R}$ which illustrates some of the properties above (in fact, it is a simple example of what is called a fractal). It will reappear later in the course.

The ("middle third") Cantor set is constructed as follows:

Step 0: Set $C_0 = [0, 1]$.

Step 1: Remove the middle third (as an open interval) of this set, leaving

$$C_1 = [0, 1/3] \cup [2/3, 1].$$

Step N: From each of 2^{N-1} closed intervals from C_{N-1} remove the open middle third to give a new set that consists of 2^N closed intervals.

Note that C_N consists of 2^N closed intervals, each of length 3^{-N} (so their total length is $(2/3)^N \to 0$, as $N \to \infty$).

The set

$$C = \bigcap_{n=0}^{\infty} C_n$$

is the (middle third) Cantor set. Since each C_n is closed, C is closed (as it is the intersection of closed sets). C is non-empty: it contains the endpoints of every interval that we remove (in fact it contains uncountably many points, which we will prove later).

The interior of C is empty: if not it would contain some open set of length $\ell > 0$; but the total length of the intervals in C_n is $(2/3)^n$ which is $< \ell$ for n large enough. So since C is closed, it is nowhere dense.

We have $C = \partial C$, since $\overline{C} = C$ and $C^\circ = \varnothing$; it also follows from this that C is nowhere dense.

The set C contains no isolated points: for any $\epsilon > 0$ any point in C was in an interval of length $< \epsilon/2$, and the two endpoints of this interval are both in C.

5.6 The Hausdorff property and metrisability

If we think of a metric space as a topological space, we use the topology that comes from the metric (unless we specify otherwise).

Definition 5.24. A topological space $(T, \mathcal{T})$ is metrisable if there is a metric d on T such that $\mathcal{T}$ consists of the open sets in (T, d).

Not all topological spaces are metrisable: we have already seen this for the indiscrete topology. But there are other (more natural) topologies that cannot be derived from a metric.

One way to show that a topology is not metrisable: find a property that any metrisable topological space must have, and show that it fails. The Hausdorff property is one such. However, before we introduce it, we need to give a definition of convergence of sequences in topological spaces.

Definition 5.25. A sequence $(x_n)_{n=1}^{\infty}$ in a topological space T converges to x if for every open neighbourhood U of x there exists $N \geq 1$ such that $x_n \in U$ for all $n \geq N$.

You should convince yourselves that if (X, d) is a metric space then convergence (in the sense of this definition) in the topology induced by the metric is just the same as convergence in (X, d) (as in Definition 3.20).

However, this definition sometimes gives rise to unexpected behaviour. For example, suppose that T is any set with the indiscrete topology. Then any sequence (x_n) in T converges to any point x in T: the only open set containing x is T itself, and $x_n \in T$ for all $n \geq 1$.

The problem is that (unlike in a metric space) we cannot separate two different points using open sets. To identify a class of topological spaces that avoid this problem, we make the following definition.

Definition 5.26. A topological space T is *Hausdorff* if for any two distinct $x, y \in T$ there exist disjoint open sets U, V such that $x \in U$, $y \in V$.

Examples:

- Any metric space is Hausdorff. Take $x, y \in X$ with $x \neq y$ and set $\epsilon = d(x, y) > 0$. Then
$$x \in B(x, \epsilon/2), \qquad y \in B(y, \epsilon/2), \qquad B(x, \epsilon/2) \cap B(y, \epsilon/2) = \varnothing.$$
  As a consequence, *any metrisable topological space must be Hausdorff.*

- The indiscrete topology is not Hausdorff (the only open set containing x is T, which also contains y), so is not metrisable.

- The co-finite topology on any infinite set is not Hausdorff: any two open sets have finite complements, so they must intersect. It follows that this topology is not metrisable.

Lemma 5.27. *In a Hausdorff space T any sequence has at most one limit.*

Proof. Suppose that $x_n \to x$ and $x_n \to y$ with $x \neq y$. Then we can find disjoint open sets U, V with $x \in U$ and $y \in V$. By the definition of convergence, there exist $N_1 \geq 1$ and $N_2 \geq 1$ such that
$$x_n \in U \ \forall n \geq N_1 \qquad \text{and} \qquad x_n \in V \ \forall n \geq N_2.$$
Taking some $N \geq \max\{N_1, N_2\}$ yields a contradiction, as $U \cap V = \varnothing$. □

There are non-Hausdorff topologies in which convergent sequences have a unique limits, see Problem Sheet 3.

5.7 Continuity between topological spaces

We have seen that we can characterise continuity between metric spaces using only open sets (Theorem 4.7). We will use the same idea to *define* continuity between topological spaces.

Definition 5.28. A map $f : T_1 \to T_2$ between two topological spaces $(T_1, \mathcal{T}_1)$ and $(T_2, \mathcal{T}_2)$ is *continuous* if whenever $U \subset T_2$ is open, $f^{-1}(U)$ is open in T_1 (i.e. if $U \in \mathcal{T}_2$ then $f^{-1}(U) \in \mathcal{T}_1$).

Examples:

- Any constant map with $f(x) = y$ for some fixed $y \in T_2$. Then $f^{-1}(U) = T_1$ if $y \in U$ and $f^{-1}(U) = \varnothing$ if $y \notin U$.

- The identity map $f : T_1 \to T_1$ (with the same topology in the domain and the image).

- Continuous maps between metric spaces (with the topology coming from the metrics).
- Any map of a space with the discrete topology to any topological space ($f^{-1}(U)$ is a subset of T_1 and so open, because any set is open in the discrete topology).
- If T_1 has the indiscrete topology then $f : T_1 \to \mathbb{R}$ is continuous if and only if it is constant (we showed continuity of constant maps; if f is not constant then we have $f(x) \neq f(y)$ for some $x, y \in T_1$, and then, taking an open set $U \subset \mathbb{R}$ that contains $f(x)$ but not $f(y)$, we have $f^{-1}(U) \supset f^{-1}(\{f(x)\}) \neq \varnothing$ (since it contains x) and $f^{-1}(U) \neq T_1$ (since it does not contain y).

To check that a map is continuous between two topological spaces it is enough to check it for a sub-basis (since any basis is also a sub-basis, we could check for a basis if we wanted).

Lemma 5.29. *Suppose that $f : T_1 \to T_2$ is a map between two topological spaces $(T_1, \mathcal{T}_1)$ and $(T_2, \mathcal{T}_2)$, and that $\mathcal{B}$ is a sub-basis for the topology $\mathcal{T}_2$. Then f is continuous if and only if $f^{-1}(B)$ is open in T_1 for every $B \in \mathcal{B}$.*

Not examinable

Proof. The "only if" direction is clear since every element of the sub-basis is an element of $\mathcal{T}_2$.

Now, any element U of $\mathcal{T}_2$ can be written as $U = \bigcup_i D_i$, where each D_i is a finite intersection of elements of $\mathcal{B}$. So

$$f^{-1}(U) = f^{-1}\left(\bigcup_i D_i\right) = \bigcup_i f^{-1}(D_i),$$

and, since for each D_i we may write $D_i = \bigcap_{j=1}^{n(i)} B_j$ with $B_j \in \mathcal{B}$, we have

$$f^{-1}\left(\bigcap_{j=1}^{n} B_j\right) = \bigcap_{j=1}^{n} f^{-1}(B_j),$$

which is open by assumption. So $f^{-1}(U)$ is a union of open sets and hence is open. □

5.8 Basic properties

We can easily give a topological version of Lemma 4.9.

Lemma 5.30. *If $(T_1, \mathcal{T}_1)$, $(T_2, \mathcal{T}_2)$, and $(T_3, \mathcal{T}_3)$ are topological spaces and $f : T_1 \to T_2$ and $g : T_2 \to T_3$ are continuous, then $g \circ f : T_1 \to T_3$ is continuous.*

Proof. If U is open in T_3, then $g^{-1}(U)$ is open in T_2 by continuity of g. So $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open in T_1 by continuity of f. □

We now discuss continuity in product spaces. Suppose that $(T_1, \mathcal{T}_1)$ and $(T_2, \mathcal{T}_2)$ are two topological spaces. We define two projections on $T_1 \times T_2$,

$$\pi_j : T_1 \times T_2 \to T_j, \quad j = 1, 2,$$

by setting

$$\pi_1(x,y) = x \qquad \text{and} \qquad \pi_2(x,y) = y.$$

Lemma 5.31. *For $j = 1, 2$, the projection $\pi_j : T_1 \times T_2 \to T_j$ is continuous (where we use the product topology on $T_1 \times T_2$).*

Proof. If $U_1 \subset T_1$ is open, then $\pi_1^{-1}(U_1) = U_1 \times T_2$, which is open. □

Lemma 5.32. *Let $(T, \mathcal{T})$, $(T_1, \mathcal{T}_1)$ and $(T_2, \mathcal{T}_2)$ be topological spaces. A map $f = (f_1, f_2) : T \to T_1 \times T_2$ is continuous if and only if f_1 and f_2 are both continuous (i.e. $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous).*

Proof. ($\Rightarrow$) Since π_j is continuous, so is $\pi_j \circ f$ (by Lemma 5.30).

Not examinable

($\Leftarrow$) By Lemma 5.29, we only have to show that $f^{-1}(B)$ is open for every set from a basis for the product topology. But the sets $U_1 \times U_2$, with U_i open in T_i, form a basis, and we have

$$f^{-1}(U_1 \times U_2) = f_1^{-1}(U_1) \cap f_2^{-1}(U_2)$$

is open in T.

□

If we consider maps from $T \to \mathbb{R}$ then we can consider sums, products and quotients of these maps.

Lemma 5.33. *If $f, g : T \to \mathbb{R}$ are continuous then so are $f + g$, fg, and f/g is continuous on the set $\{x \in T : g(x) \neq 0\}$.*

Not examinable

Proof. We give the argument for $f+g$, noting that the intervals (a, ∞) and $(-\infty, b)$ (for every $a, b \in \mathbb{R}$) are a sub-basis for the topology of $\mathbb{R}$. So it is enough to show that $(f+g)^{-1}((a,\infty))$ and $(f+g)^{-1}((-\infty, b))$ are open in T, using Lemma 5.29. We have

$$\begin{aligned} f(x) + g(x) > a \quad &\Leftrightarrow \quad f(x) > a - g(x) \\ &\Leftrightarrow \quad f(x) > r \text{ and } r > a - g(x) \quad \text{for some } r \\ &\Leftrightarrow \quad f(x) > r \text{ and } g(x) > a - r \quad \text{for some } r. \end{aligned}$$

It follows that

$$\{x : f(x) + g(x) > a\} = \bigcup_{r \in \mathbb{R}} \{x : f(x) > r\} \cap \{x : g(x) > a - r\},$$

which is open. Similarly, we can show that

$$f^{-1}((-\infty, b)) = \{x : f(x) + g(x) < b\}$$

is open. □

Alternative proof: The function $\sigma : \mathbb{R}^2 \to \mathbb{R}$ given by $\sigma(x, y) = x + y$ is continuous from $\mathbb{R}^2$

to $\mathbb{R}$; it is in fact Lipschitz continuous from $(\mathbb{R}^2, \varrho_1)$ into $\mathbb{R}$, since

$$d_{\mathbb{R}}(x_1 + y_1, x_2 + y_2) = |(x_1 + y_1) - (x_2 + y_2)| \\ \leq |x_1 - x_2| + |y_1 - y_2| \leq \varrho_1((x_1, y_1), (x_2, y_2)).$$

The map $x \mapsto (f(x), g(x))$ is continuous from T into $\mathbb{R}^2$ by Lemma 5.32. So the composition $(\sigma \circ (f, g))(x) = f(x) + g(x)$ is continuous from T into $\mathbb{R}$. A similar argument works for products and quotients.

Example: The map from $\mathbb{R}^2 \to \mathbb{R}^2$ given by

$$(x, y) \mapsto (x + y, \sin(x^2 y^3))$$

is continuous. We know that $(x, y) \mapsto x$ and $(x, y) \mapsto y$ are both continuous (they are projections - Lemma 5.31), as are the maps $(x, y) \mapsto x + y$ [sums of continuous functions are continuous], $(x, y) \mapsto x^2 y^3$ [products of continuous functions are continuous], and $(x, y) \mapsto \sin(x^2 y^3)$ [composition with the continuous function $t \mapsto \sin(t)$]. Since both components are continuous, the whole map is continuous (Lemma 5.31 again).

Not examinable

5.9 The projective topology and product spaces

Consider a set T (without a topology), a collection of topological spaces $(T_j, \mathcal{T}_j)$ and a collection of maps $f_j : T \to T_j$, where j is in some arbitrary indexing set $\mathcal{J}$. This data allows us to define a topology on T.

Definition 5.34. The *projective topology* on T is the coarsest topology for which all the maps $f_j : T \to T_j$ are continuous.

Let us unpick this definition. Recall that the "coarsest topology" is the one with the smallest collection of open sets. In order for f_j to be continuous, we must have that $f_j^{-1}(U)$ is open whenever $U \in \mathcal{T}_j$. So the projective topology contains

$$\mathcal{B} := \bigcup_{j \in \mathcal{J}} \{f_j^{-1}(U) : U \in \mathcal{T}_j\}.$$

This is not a basis for a topology since if $B_1, B_2 \in \mathcal{B}$ then $B_1 \cap B_2$ is not necessarily a union of sets in $\mathcal{B}$. However, the union of sets in $\mathcal{B}$ is equal to T (since, for any j, $f_j^{-1}(T_j) = T$), so we can apply Proposition 5.9 to get that the projective topology is the unique topology with $\mathcal{B}$ as a sub-basis.

An example of this way of obtaining a topology is given by the product topology defined in the previous section. Let $(T_1, \mathcal{T}_1)$ and $(T_2, \mathcal{T}_2)$ be topological spaces. Recall that the product topology $\mathcal{T}$ on $T_1 \times T_2$ is the topology with basis

$$\{U_1 \times U_2 : U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2\}.$$

We claim that the product topology $\mathcal{T}$ on $T_1 \times T_2$ may also be characterised as the coarsest topology for which the two projection maps $\pi_j : (T_1 \times T_2) \to T_j$ $(j = 1, 2)$ are continuous, which we'll call $\mathcal{T}'$. We need to show $\mathcal{T}' = \mathcal{T}$. First note that, for any $U_1 \in \mathcal{T}_1$, $\mathcal{T}'$ must contain $\pi_1^{-1}(U_1) = U_1 \times T_2$ and, similarly, for any $U_2 \in \mathcal{T}_2$, $\mathcal{T}'$ must contain $\pi_2^{-1}(U_2) = T_1 \times U_2$. So $\mathcal{T}'$ must contain the intersection of such sets, i.e. $U_1 \times U_2$. In other words, $\mathcal{T}' \supset \mathcal{B}$ and therefore $\mathcal{T}' \supset \mathcal{T}$. On the other hand, by definition, for $U_1 \in \mathcal{T}_1$, $\mathcal{T}$ contains $U_1 \times T_2 = \pi_1^{-1}(U_1)$ and, for $U_2 \in \mathcal{T}_2$, $\mathcal{T}$ contains $T_1 \times U_2 = \pi_2^{-1}(U_2)$. So, $\mathcal{T}$ is a topology that makes π_1 and π_2 continuous. Since $\mathcal{T}'$ is the coarsest such topology, we have $\mathcal{T} \supset \mathcal{T}'$. Hence $\mathcal{T} = \mathcal{T}'$, as required.

We will use this approach to define the product topology for an arbitrary product.

Definition 5.35. Let $(T_j, \mathcal{T}_j)$, $j \in J$, be an arbitrary collection of topological spaces. Their product $T = \prod_{j \in J} T_j$ is the set of all functions $x : J \to \bigcup_{j \in J} T_j$ such that $x(j) \in T_j$ (see Appendix 9.2 for more details). The product topology $\mathcal{T}$ on T is the coarsest topology for which all of the projections

$$\pi_j : T \to T_j : x \mapsto x(j)$$

are continuous. We then call the topological space $(T, \mathcal{T})$ the topological product of the spaces $(T_j, \mathcal{T}_j)$.

A sub-basis for the product topology consists of all sets of the form

$$\prod_{j \in J} U_j,$$

where $U_j \in \mathcal{T}_j$ with $U_j = T_j$ except for a finite number of the j.

5.10 Homeomorphisms

We have already defined the notion of a homeomorphism between two metric spaces. The definition between two topological spaces is essentially the same.

Definition 5.36. Let $(T_1, \mathcal{T}_1)$ and $(T_2, \mathcal{T}_2)$ be topological spaces. A bijection $f : T_1 \to T_2$ is a *homeomorphism* if any one of the following equivalent conditions holds:

(i) both f and f^{-1} are continuous;

(ii) V is open in T_2 if and only if $f^{-1}(V)$ is open in T_1;

(iii) U is open in T_1 if and only if $f(U)$ is open in T_2.

If there is a homeomorphism $f : T_1 \to T_2$ we say that $(T_1, \mathcal{T}_1)$ and $(T_2, \mathcal{T}_2)$ are *homeomorphic*.

Note: There are continuous bijections that are not homeomorphisms, for example, let T_1 be $[0, 1]$ with the standard topology, T_2 be $[0, 1]$ with the indiscrete topology, and consider the identity map from T_1 to T_2. Then the only open sets in T_2 are $\varnothing$ and $[0, 1]$, and these are also open in T_1 (so that identity map is continuous from T_1 to T_2); but there are many open sets in T_1 that are not open in T_2 (so the inverse – also the identity – is not continuous).

Definition 5.37. A property of topological spaces is a *topological invariant* (or 'topological property') if it is preserved by homeomorphisms.

Examples:

- T is finite;
- T is Hausdorff;
- T is metrisable;
- every continuous real-valued function on T is bounded.

To show that two sets are not homeomorphic, we need to find a *topological property* that one has and the other does not. For example, $[0, 1]$ and $\mathbb{R}$ are not homeomorphic - every continuous function real-valued on $[0, 1]$ is bounded, but this is not true on $\mathbb{R}$.

6 Compactness

6.1 Definition and the Heine–Borel Theorem

Definition 6.1. A *cover* of a set A is collection $\mathcal{U}$ of sets whose union contains A:

$$A \subset \bigcup_{U \in \mathcal{U}} U.$$

A *subcover* of a cover $\mathcal{U}$ is a subset of $\mathcal{U}$ whose elements still cover A. A cover is *open* if all of its elements are open.

Examples:

- $\{(n, n+3) : n \in \mathbb{Z}\}$ is an open cover of $\mathbb{R}$; $\{(2k, 2k+3) : k \in \mathbb{Z}\}$ is a subcover;
- $\{(n, n+1) : n \in \mathbb{Z}\}$ is not an open cover of $\mathbb{R}$ [since it does not contain the integers);
- $\{(a-1, a+1) : a \in \mathbb{R}\}$ is a cover of $\mathbb{R}$; $\{(a-1, a+1) : a \in \mathbb{Z}\}$ is a subcover; $\{(2a-1, 2a+1) : a \in \mathbb{Z}\}$ is not a subcover (since it misses out all odd integers) and $\{(a-1/2, a+1/2) : a \in \mathbb{R}\}$ is not a subcover (it does cover $\mathbb{R}$, but it is not a subcollection of the original cover).

Definition 6.2. A topological space T is *compact* if every open cover of T has a finite subcover.

Examples of non-compact spaces:

- $(0, 1)$ is not compact - $\{(0, a) : a \in (0, 1)\}$ is an open cover with no finite subcover;
- $\mathbb{R}$ is not compact - $\{(-\infty, a) : a \in \mathbb{Z}\}$ has no finite subcover.

Definition 6.3. A subset S of T is compact if every open cover of S by subsets of T has a finite subcover. This is the same as S being compact with the subspace topology.

Lemma 6.4. *If T is a topological space and $S \subset T$ then S is compact in the sense of Definition 6.3 if and only if $(S, \mathcal{T}_S)$ is compact in the sense of Definition 6.2.*

Proof. Suppose that S is a compact subset of T (using Definition 6.3). Any open cover $\mathcal{U}$ of $(S, \mathcal{T}_S)$ consists of sets of the form $U = V(U) \cap S$ where $V(U)$ is open in T; so

$$S \subset \bigcup_{U \in \mathcal{U}} U = \bigcup_{U \in \mathcal{U}} V(U) \cap S \subset \bigcup_{U \in \mathcal{U}} V(U);$$

since this is an open cover of S by open sets in T it has a finite subcover:

$$S \subset \bigcup_{j=1}^{n} V(U_j).$$

Now we have

$$S \subset \bigcup_{j=1}^{n} V(U_j) \cap S = \bigcup_{j=1}^{n} U_j,$$

i.e. the original cover has a finite subcover.

If $(S, \mathcal{T}_S)$ is compact (using Definition 6.2) and $\mathcal{U}$ is an open cover of S (using sets in $\mathcal{T}$) then

$$S \subset \bigcup_{U \in \mathcal{U}} U \qquad \Rightarrow \qquad S \subset \bigcup_{U \in \mathcal{U}} U \cap S;$$

this gives an open cover of S by sets in $\mathcal{T}_S$, so this has a finite subcover,

$$S \subset \bigcup_{j=1}^{n} U_j \cap S \subset \bigcup_{j=1}^{n} U_j,$$

so $\mathcal{U}$ also has a finite subcover. □

The most basic compactness theorem is the Heine–Borel Theorem.

Theorem 6.5 (Heine–Borel Theorem)**.** *Any closed interval* $[a, b]$ *is a compact subset of* $\mathbb{R}$ *(with the usual topology).*

We will allow the notation $[x, x] = \{x\}$ in the proof.

Proof. Let $\mathcal{U}$ be a cover of $[a, b]$ by open subsets of $\mathbb{R}$.

Let A denote the set all of $x \in [a, b]$ such that $[a, x]$ can be covered by a finite subcover taken from $\mathcal{U}$. Since $a \in A$ (we can certainly cover $[a, a] = \{a\}$ by one element of $\mathcal{U}$) the set A is non-empty.

The set A is also bounded above by b, so we can set $c = \sup A$. Since $a \leq c \leq b$, we must have $c \in U$ for some $U \in \mathcal{U}$. Since U is open, we have $(c - \delta, c + \delta) \subset U$ for some $\delta > 0$.

Since $c = \sup A$, there exists some $x \in A$ with $x > c - \delta$. It follows that

$$[a, c + \delta) = [a, x] \ \cup \ (c - \delta, c + \delta)$$

can also be covered by a finite collection of sets from $\mathcal{U}$, since $[a, x]$ can be and $(c - \delta, c + \delta) \subset U \in \mathcal{U}$.

It follows (i) that $c = b$, for otherwise $c < b$ and this yields a cover of

$$[a, \min(c + \delta/2, b)]$$

by a finite number of sets from $\mathcal{U}$, contradicting the fact that $c = \sup A$; so (ii) a finite collection of sets from $\mathcal{U}$ cover $[a, b + \delta) \supset [a, b]$, which is what we wanted. □

6.2 Compact vs closed

We now investigate the relationship between compactness and being closed/bounded.

Lemma 6.6. *Any closed subset* S *of a compact space* T *is compact.*

Proof. Let $\mathcal{U}$ be a cover of S by open subsets of T. Then $\mathcal{U} \cup \{T \setminus S\}$ is an open cover of T, so has a finite subcover; elements of this subcover (removing $T \setminus S$ if it is included) provide a finite open subcover of S. □

Lemma 6.7. *Any compact subset K of a Hausdorff space T is closed.*

Proof. Suppose that $a \in T \setminus K$. For each $x \in K$ there exist disjoint open sets $U_x \ni x$ and $V_x \ni a$. The open sets $\{U_x : x \in K\}$ form an open cover of K, so there is a finite subcover $U_{x_1}, \ldots, U_{x_n}$ of K. Then

$$V = \bigcap_{i=1}^{n} V_{x_i}$$

is an open set that contains a that is disjoint from K. Thus $T \setminus K$ is open. Since $T \setminus K$ is open, K is closed. □

Note that the result need not be true if T is not Hausdorff. Consider any set T containing at least two points with the indiscrete topology. Then any subset S of T is compact, since the only open covers of S that are available available are $\{\varnothing, T\}$ and $\{T\}$, which always have a finite subcover. But S is not closed unless $S = \varnothing$ or $S = T$.

Lemma 6.8. *Any compact subset K of a metric space (X, d) is bounded.*

Proof. Choose any $a \in X$. If $x \in K$ then $x \in B(a, r)$ for all $r > d(a, x)$. Hence K is covered by the collection of open balls $\{B(a, r) : r > 0\}$, so there is a finite subcover

$$\{B(a, r_i) : i = 1, \ldots, n\},$$

so

$$K \subset \bigcup_{i=1}^{n} B(a, r_i) = B(a, \max_i r_i),$$ □

giving that K is bounded.

Corollary 6.9. *A subset of $\mathbb{R}$ (with the usual topology) is compact if and only if it is closed and bounded.*

Proof. Since $\mathbb{R}$ is a metric space, any compact subset is bounded; since it is Hausdorff, any compact subset is closed.

If $K \subset \mathbb{R}$ is bounded then $K \subset [-R, R]$ for some $R > 0$. Then K is a closed subset of the compact set $[-R, R]$, so is compact. □

Theorem 6.10. *Let $\mathcal{F}$ be a collection of non-empty closed subsets of a compact space T such that every finite subcollection of $\mathcal{F}$ has a non-empty intersection. Then the intersection of all the sets from $\mathcal{F}$ is non-empty.*

Proof. Suppose that the intersection of all the sets from $\mathcal{F}$ is empty, and let $\mathcal{U}$ be the collection of their complements,

$$\mathcal{U} := \{T \setminus F : F \in \mathcal{F}\}.$$

Then $\mathcal{U}$ is an open cover of T, since

$$T \setminus \bigcup_{U \in \mathcal{U}} U = \bigcap_{U \in \mathcal{U}} (T \setminus U) = \bigcap_{F \in \mathcal{F}} F = \varnothing.$$

Thus $\mathcal{U}$ has a finite subcover $U_1, \ldots, U_n$, which implies that

$$\varnothing = T \setminus \bigcup_{i=1}^{n} U_i = \bigcap_{i=1}^{n} F_i,$$

a contradiction. □

One can also reverse this proof: if T is a topological space such that this result holds then T is compact (see Problem Sheet 6).

Corollary 6.11. *Let $F_1 \supset F_2 \supset F_3 \supset \cdots$ be non-empty closed subsets of a compact space T. Then $\bigcap_{j=1}^{\infty} F_j \neq \varnothing$.*

6.3 Compactness of products and compact subsets of $\mathbb{R}^n$

We now show that products of compact spaces are compact.

Theorem 6.12. *If T and S are compact topological spaces then $T \times S$ is compact.*

We write $\mathcal{T}$ for the topology on T and $\mathcal{S}$ for the topology on S. Recall that the product topology on $T \times S$ is the topology with basis

$$\{U \times V : \ U \in \mathcal{T}, \ V \in \mathcal{S}\},$$

i.e. open sets in $T \times S$ are formed of unions of sets of the form $U \times V$. It follows that if $(t, s) \in W \subset T \times S$, with W an open set in $T \times S$, there exist $U \in \mathcal{T}$ and $V \in \mathcal{S}$ such that

$$(t, s) \in U \times V \subset W.$$

Proof. Suppose that $\mathcal{U}$ is an open cover of $T \times S$.

We first show the following claim.

Claim: For each $s \in S$ there is an open set $N(s) \subset S$ with $s \in N(s)$ [an open neighbourhood of s in S] such that $T \times N(s)$ can be covered by a finite subfamily of $\mathcal{U}$.

Justification: For each $x \in T$, we can find $W_x \in \mathcal{U}$ such that $(x, s) \in W_x$. By the definition of the product topology (see above) there exist $U_x \in \mathcal{T}$, $V_x \in \mathcal{S}$ such that

$$(x, s) \in U_x \times V_x \subset W_x.$$

The sets U_x form an open cover of T, so they contain a finite subcover $U_{x_1}, \ldots U_{x_n}$. If we let

$$N(s) = \bigcap_{i=1}^{n} V_{x_i}$$

then $N(s) \subset S$ is open, $s \in N(s)$ (so it is not empty), and

$$T \times N(s) \subset \bigcup_{i=1}^{n} (U_{x_i} \times V_{x_i}) \subset \bigcup_{i=1}^{n} W_{x_i}.$$

This proves the claim.

We now complete the proof of the theorem. The family $\{N(s) : s \in S\}$ forms an open cover of S, so there is a finite subcover $\{N(s_1), \ldots, N(s_n)\}$ that covers S. Thus

$$T \times S = \bigcup_{j=1}^{n} T \times N(s_j).$$

This is a finite union, and, by the claim, each of the sets $T \times N(s_j)$ can be covered by a finite subfamily of $\mathcal{U}$, so $T \times S$ can be covered by a finite subfamily of $\mathcal{U}$. □

The following corollary is immediate.

Corollary 6.13. *The product of a finite number of compact spaces is compact.*

The following result, however, involves some quite powerful mathematical machinery, and we will not prove it here.

Theorem 6.14 (Tychonov's Theorem)**.** *The product of any collection of compact spaces is compact (with the product topology).*

As an application of Corollary 6.13, we can characterise the compact subsets of $\mathbb{R}^n$.

Theorem 6.15 (Heine–Borel in $\mathbb{R}^n$)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. Let K be a compact subset of $\mathbb{R}^n$. Any metric space is Hausdorff: so it follows from Lemma 6.7 that K is closed, and from Lemma 6.8 that K is bounded.

For the converse, observe that if K is bounded then $K \subset [-R, R]^n$ for some $R > 0$. Since $[-R, R]$ is compact, it follows from Theorem 6.12 that $[-R, R]^n$ is compact. Now K, as a closed subset of $[-R, R]^n$, must be compact, using Lemma 6.6. □

Note, however, that in general metric spaces closed and bounded does not imply compact. For example, the set $(0, 1)$ in the metric space $(0, 1)$ is closed and bounded, but not compact.

6.4 Continuous functions on compact sets

Theorem 6.16. *A continuous image of a compact space is compact.*

Proof. Let T be compact and $f : T \to S$ continuous. Let $\mathcal{U}$ be an open cover of $f(T)$; since f is continuous, the sets $f^{-1}(U)$, $U \in \mathcal{U}$, are open, and form a cover of T. [For every $x \in T$, $f(x) \in f(T)$, so $f(x) \in U$ for some $U \in \mathcal{U}$.] Since T is compact, there is a finite subcover of T, $f^{-1}(U_1), \ldots, f^{-1}(U_n)$.

Now, every $y \in f(T)$ is given by $y = f(x)$ for some $x \in X$. We have $x \in f^{-1}(U_j)$ for some j, so $y = f(x) \in U_j$. So $U_1, \ldots, U_n$ are a cover of $f(T)$. □

This shows that compactness is a topological property (which, one might argue, was clear already from its definition).

Theorem 6.17. *A continuous bijection of a compact space T onto a Hausdorff space S is a homeomorphism.*

We use Corollary 4.8, which also works in a topological space: a function $f : T \to S$ is continuous if and only if $f^{-1}(V)$ is closed in T whenever V is closed in S.

Proof. We need to show that $f^{-1} : S \to T$ is continuous; since f is a bijection we have $(f^{-1})^{-1}(K) = f(K)$ for any $K \in T$.

Take any closed K in T. Then K is compact by Lemma 6.6. It follows that $f(K)$ is compact in S by the previous theorem. Since S is Hausdorff, $f(K)$ is closed (Lemma 6.7). □

Example. Let $f : [0,1] \to \mathbb{R}^n$ be an injective continuous function (so this is an injective continuous curve). Then $f([0,1])$ is homeomorphic to $[0,1]$.
(Indeed, $[0,1]$ is compact, and $\mathbb{R}^n$ (and all metric spaces) are Hausdorff, and its subsets are also Hausdorff, so Theorem 6.17 can be applied with $T = [0,1]$ and $S = f([0,1])$.)

Not examinable

We know that $f : T \to \mathbb{R}$ is continuous if the preimage of every open set is open. For functions from $T \to \mathbb{R}$ we can split 'continuity' into two parts, which also provides a quick proof of an extremely important result that generalises the Extreme Value Theorem.

Definition 6.18. A function $f : T \to \mathbb{R}$ is *lower semicontinuous* if for every $c \in \mathbb{R}$ the set $f^{-1}(c, \infty)$ is open. It is *upper semicontinuous* if for every $c \in \mathbb{R}$ the set $f^{-1}(-\infty, c)$ is open.

Observe that if $f : T \to \mathbb{R}$ is both upper and lower semicontinuous then

$$f^{-1}(a, \infty) \cap f^{-1}(-\infty, b) = f^{-1}(a, b)$$

is open for every $a, b \in \mathbb{R}$, and so f is continuous. (The preimage of every set in a basis for the topology on $\mathbb{R}$ is open, and so f is continuous by Lemma 5.29.)

Theorem 6.19. *If T is non-empty and compact and $f : T \to \mathbb{R}$ is lower semicontinuous then it is bounded below and attains its minimum. If f is upper semicontinuous then it is bounded above and attains its maximum.*

Proof. Let $c = \inf_{x \in T} f(x)$. Suppose that f is lower semicontinuous and does not attain the value c; this certainly occurs if $c = -\infty$.

In this case $f(x) > c$ for every $x \in T$, and so the open sets

$$\{x : f(x) > r\} = f^{-1}(r, \infty), \quad r > c,$$

cover T. Since T is compact, there is a finite subcover by sets

$$\{x : f(x) > r_j\}, \ j = 1, \ldots, n.$$

But the union of these sets is $\{x : f(x) > \min_j r_j\}$, and $\min_j r_j > c$, contradicting the definition of c. □

Corollary 6.20. *If T is non-empty and compact then a continuous function $f : T \to \mathbb{R}$ is bounded and attains its bounds.*

Recall that a real-valued function $f : T \to \mathbb{R}$ is bounded if there is a real number C such that $|f(x)| \le C$ for all $x \in T$, equivalently, $f(T)$ is a bounded subset of $\mathbb{R}$.

A real-valued function $f : T \to \mathbb{R}$ attains its bounds if it attains its maximum and minimum, that is, there are $x_{\min}, x_{\max} \in X$ such that

$$f(x_{\min}) = \inf_{x\in T} f(x)$$

$$f(x_{\max}) = \sup_{x\in T} f(x).$$

(For example, the function $\arctan x$ on $\mathbb{R}$ is bounded but does not attain its bounds.)

Direct proof of Corollary 6.20. Since T is compact and $f : T \to \mathbb{R}$ is continuous, $f(T)$ is a compact subset of $\mathbb{R}$. So $f(T)$ is closed and bounded. Now note that any closed and bounded subset F of $\mathbb{R}$ contains its supremum: for each $n \in \mathbb{N}$ there exists $x_n \in F$ such that

$$\sup F - \frac{1}{n} < x_n \le \sup F,$$

so $x_n \to \sup F$, and since F is closed we have $\sup F \in F$.

Therefore $f(T)$ contains its supremum. It follows that there exists $x_{\max} \in T$ such that $f(x_{\max}) = \sup_{x\in T} f(x)$. We can argue similarly for the infimum. □

6.5 Equivalence of all norms on $\mathbb{R}^n$

We now use Corollary 6.20 to show that all norms on $\mathbb{R}^n$ are equivalent, i.e. given any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{R}^n$ there are constants $0 < c_1 \le c_2$ such that

$$c_1\|x\|_1 \le \|x\|_2 \le c_2\|x\|_1 \qquad \text{for every } x \in \mathbb{R}^n.$$

This means that there is only one topology on $\mathbb{R}^n$ that comes from a norm (the 'standard one').

Theorem 6.21. *All norms on $\mathbb{R}^n$ are equivalent.*

Not examinable

Proof. We show that any norm $\|\cdot\|$ is equivalent to the standard 'Euclidean' norm $\|\cdot\|_{\ell^2}$. Let $(e_j)_{j=1}^n$ be an orthonormal basis for $\mathbb{R}^n$, so that any $x \in \mathbb{R}^n$ can be written

$$x = \sum_{j=1}^n x_j e_j.$$

The standard norm is given by

$$\|x\|_{\ell^2}^2 = \sum_{j=1}^n |x_j|^2.$$

Now we have

$$\begin{aligned}\|x\| = \left\|\sum_{j=1}^n x_j e_j\right\| &\le \sum_{j=1}^n |x_j|\|e_j\| \\ &\le \left(\sum_{j=1}^n |x_j|^2\right)^{1/2} \left(\sum_{j=1}^n \|e_j\|^2\right)^{1/2} \\ &= c_2\|x\|_{\ell^2},\end{aligned}$$

where $c_2 = \left(\sum_{j=1}^n \|e_j\|^2\right)^{1/2}$.

Replacing x by $x - y$ this yields $\|x-y\| \le c_2\|x-y\|_{\ell^2}$, and so the map $x \mapsto \|x\|$ is continuous from $(\mathbb{R}^n, \|\cdot\|_{\ell^2})$ to $\mathbb{R}$. Now, the unit sphere

$$S := \{x \in \mathbb{R}^n : \|x\|_{\ell^2} = 1\}$$

is a closed bounded subset of $\mathbb{R}^n$, so is compact. It follows that the continuous map $x \mapsto \|x\|$ is bounded and attains its bounds. In particular, it is bounded below: if $\|x\|_{\ell^2} = 1$ then $\|x\| \ge c_1$ for some $c_1 \ge 0$. We must have $c_1 > 0$, since the function attains its bounds; if $c_1 = 0$ then there would be some $x \in S$ with $\|x\| = 0$. Since $\|\cdot\|$ is a norm this would imply that $x = 0$; but we know that $\|x\|_{\ell^2} = 1$, so this is impossible.

It follows that $\|x\|_{\ell^2} = 1$ implies that $\|x\| \ge c_1$. Now for any non-zero $y \in \mathbb{R}^n$ we can set $x = y/\|y\|_{\ell^2}$ and then

$$\|x\|_{\ell^2} = 1 \quad \Rightarrow \quad \left\|\frac{y}{\|y\|_{\ell^2}}\right\| \ge c_1 \quad \Rightarrow \quad \|y\| \ge c_1\|y\|_{\ell^2}.$$ □

Note: There are many non-equivalent *metrics* on $\mathbb{R}^n$, e.g. the discrete metric is not equivalent to any of the ℓ^p metrics.

6.6 Lebesgue numbers and uniform continuity

Definition 6.22. Let $\mathcal{U}$ be an open cover of a metric space (X, d). A number $\delta > 0$ is called a *Lebesgue*[2] *number* for $\mathcal{U}$ if for any $x \in X$ there exists $U \in \mathcal{U}$ such that $B(x, \delta) \subset U$.

In general an open cover will not have a Lebesgue number. For example the sets $(x/2, x)$, $x \in (0, 1)$, form an open cover of $(0, 1)$ with no Lebesgue number.

Proposition 6.23. *Every open cover $\mathcal{U}$ of a compact metric space (X, d) has a Lebesgue number.*

Not examinable

First proof. For each $x \in X$ let

$$r(x) = \sup\{r \in (0, 1] : B(x, r) \subset U, \text{ for some } U \in \mathcal{U}\}.$$

Note that $r(x) > 0$ for every x, since if $x \in X$ then $x \in U$ for some U, and since U is open there exists $r > 0$ such that $B(x, r) \subset U$.

If we can show that $r : X \to \mathbb{R}$ is lower semicontinuous then we can use Theorem 6.19 to show that r is bounded below and attains its bound, and then $\delta := (\inf_{x \in X} r(x))/2 > 0$ is a Lebesgue number of $\mathcal{U}$.

So take $c \in \mathbb{R}$ and consider

$$W = r^{-1}(c, \infty) = \{y \in X : r(y) > c\};$$

we need to show that W is open. Take $x \in W$, set $\epsilon = (r(x) - c)/3$, and find $U \in \mathcal{U}$ such that

$$B(x, r(x) - \epsilon) \subset U$$

(this is possible since $B(x, r) \subset U$ for some $U \in \mathcal{U}$ for any $r < r(x)$). It follows that if $d(y, x) < \epsilon$ then

$$B(y, c + \epsilon) \subset B(x, c + 2\epsilon) = B(x, r(x) - \epsilon) \subset U,$$

so $r(y) \geq c + \epsilon > c$; it follows that $B(x, \epsilon) \subset W$, so W is open. □

Second proof. For every $x \in X$ there exists $r(x) > 0$ such that

$$B(x, r(x)) \subset U(x) \qquad \text{for some } U(x) \in \mathcal{U}.$$

The collection $\{B(x, r(x)/2) : x \in X\}$ forms an open cover of X, so has a finite subcover

$$\{B(x_j, r(x_j)/2) : j = 1, \ldots, n\}.$$

Set $\delta = \min_j r(x_j)/2$; we claim that δ is a Lebesgue number for $\mathcal{U}$.

Given any $x \in X$, we must have $x \in B(x_j, r(x_j)/2)$ for some j. Then, since $\delta \leq r(x_j)/2$ for all j,

$$B(x, \delta) \subset B(x_j, \frac{r(x_j)}{2} + \delta) \subset B(x_j, r(x_j)) \subset U(x_j);$$

so δ is a Lebesgue number for $\mathcal{U}$. □

[2]Named after the French mathematician Henri Lebesgue, 1875–1941, who created Lebesgue theory of integration, seen in the module MA359 Measure Theory.

We can use this to give a quick proof of the uniform continuity of continuous maps on compact metric spaces.

Definition 6.24. A map $f : (X, d_X) \to (Y, d_Y)$ is *uniformly continuous* if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \qquad \Rightarrow \qquad d_Y(f(x), f(y)) < \epsilon$$

for any $x, y \in X$.

Note that the key point of the definition is that δ does not depend on x or y.

Theorem 6.25. *A continuous map from a compact metric space into a metric space is uniformly continuous.*

Proof. Let $f : X \to Y$ be continuous and choose $\epsilon > 0$. For $z \in X$, define

$$U_z = f^{-1}(B_Y(f(z), \epsilon/2)).$$

Then the sets U_z, $z \in X$, form an open cover $\mathcal{U}$ of X. Let δ be a Lebesgue number of this cover. Then if $x, y \in X$ and $d_X(x, y) < \delta$ we have $y \in B(x, \delta)$; by the definition of δ, there is an element U_z of $\mathcal{U}$ such that $B(x, \delta) \subset U_z$. But then

$$d_Y(f(x), f(y)) \leq d_Y(f(x), f(z)) + d_Y(f(z), f(y)) < \epsilon.$$ □

6.7 Sequential compactness

Definition 6.26. A subset K of a metric space (X, d) is *sequentially compact* if every sequence in K has a convergent subsequence whose limit lies in K.

Note we can take $K = X$ in the definition. We want to show that compactness and sequential compactness in a metric space are equivalent. First we need a lemma guaranteeing the existence of a Lebesgue number of an open cover of a *sequentially compact* set. (We already know that there is one when the set is compact.)

Lemma 6.27. *If K is a sequentially compact subset of a metric space then any open cover of K has a Lebesgue number.*

Proof. Suppose that $\mathcal{U}$ is an open cover of K that does not have a Lebesgue number. Then for every $\epsilon > 0$ there exists $x \in K$ such that $B(x, \epsilon)$ is not contained in any element of $\mathcal{U}$.

Choose x_n such that $B(x_n, 1/n)$ is not contained in any element of $\mathcal{U}$.

Then x_n has a convergent subsequence, $x_{n_j} \to x$. Since $\mathcal{U}$ covers K, $x \in U$ for some element $U \in \mathcal{U}$. Since U is open, $B(x, \epsilon) \subset U$ for some $\epsilon > 0$.

But now take j sufficiently large that $d(x_{n_j}, x) < \epsilon/2$ and $1/n_j < \epsilon/2$. Then $B(x_{n_j}, 1/n_j) \subset B(x, \epsilon) \subset U$, contradicting the definition of x_{n_j}. □

Theorem 6.28. *A subset of a metric space is sequentially compact if and only if it is compact.*

Recall that if a sequence (x_n) converges then it must be Cauchy: for every $\epsilon > 0$ there exists N such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$ (to prove this observe that if $x_n \to x$ then there exists N such that $d(x_n, x) < \epsilon/2$ for all $n \geq N$; then $n, m \geq N$ implies that $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \epsilon$).

Proof. Step 1: Compactness implies sequential compactness.

Let (x_j) be a sequence in a compact set K. Consider the sets F_n defined by setting

$$F_n = \overline{\{x_n, x_{n+1}, \ldots\}}.$$

The sets F_n are a decreasing sequence of closed subsets of K, so we can find

$$x \in \bigcap_{j=1}^{\infty} F_j.$$

We now show that there is a subsequence that converges to x:

- since $x \in \overline{\{x_j : j \geq 1\}}$ there exists j_1 such that $d(x_{j_1}, x) < 1$;
- since $x \in \overline{\{x_j : j > j_1\}}$ there exists $j_2 > j_1$ such that $d(x_{j_2}, x) < 1/2$;
- continue in this way to find $j_k > j_{k-1}$ such that $d(x_{j_k}, x) < 1/k$.

Then x_{j_k} is a subsequence of (x_j) that converges to x. So K is sequentially compact.

Step 2: Sequential compactness implies compactness.

First we show that for every $\epsilon > 0$ there is a cover of K by a finite number of sets of the form $B(x_j, \epsilon)$ for some $x_j \in K$.

Suppose that this is not true, and that we have found points $\{x_1, \ldots, x_n\}$ such that $d(x_i, x_j) \geq \epsilon$ for all $i, j = 1, \ldots, n$. Since the collection $B(x_j, \epsilon)$ does not cover K, there exists $x_{n+1} \in K$ such that the points $\{x_1, \ldots, x_{n+1}\}$ all satisfy $d(x_i, x_j) \geq \epsilon$ for all $i, j = 1, \ldots, n+1$.

Now the sequence (x_j) has no Cauchy subsequence, so no convergent subsequence, a contradiction.

Now given any open cover $\mathcal{U}$ of K consider the finitely many points $y_1, \ldots, y_N$ such that $B(y_i, \delta)$ cover K, where δ is the Lebesgue number of the original cover. Then $B(y_i, \delta) \subset U_i$ for some $U_i \in \mathcal{U}$, and we have

$$K \subset \bigcup_{i=1}^{N} B(y_i, \delta) \subset \bigcup_{i=1}^{N} U_i,$$

so we have found a finite subcover. □

Remark: One can define sequentially compactness for an arbitrary topological space: a space $(T, \mathcal{T})$ is sequentially compact if every sequence in T has a convergent subsequence whose limit lies in T. However, then compactness and sequential compactness are not necessarily equivalent. However, examples of spaces where they are not equivalent go beyond the scope of this course.

6.8 Normed spaces

Note that the equivalence of compactness and sequential compactness in any metric space shows that these concepts are also equivalent in any normed space.

It is now easy to show that there are closed bounded subsets in general normed spaces that are not compact.

Example: the closed unit ball in ℓ^p is not compact for any $1 \leq p \leq \infty$. Consider the sequence $(\mathbf{e}^{(j)})_{j=1}^{\infty}$. Then this has no convergent subsequence: any such subsequence would have to be Cauchy, but

$$\|\mathbf{e}^{(j)} - \mathbf{e}^{(k)}\|_{\ell^p} = \begin{cases} 2^{1/p} & 1 \leq p < \infty \\ 1 & p = \infty. \end{cases}$$

In fact more is true.

Theorem 6.29. *A normed space is finite-dimensional if and only if its closed unit ball is compact.*

We will not prove this here, but see Functional Analysis I.

7 Connectedness

7.1 Definitions of connected/disconnected

Definition 7.1. We say that a pair of sets (A, B) is a partition of a topological space T if $T = A \cup B$ and $A \cap B = \varnothing$; and we then say that A and B partition T.

Note that if two open sets A and B partition T then A and B are also both closed.

Definition 7.2. A topological space T is *connected* if the only partitions of T into open sets are $(T, \varnothing)$ and $(\varnothing, T)$. The space T is said to be *disconnected* if it is not connected.

Lemma 7.3. *The following are equivalent:*

(i) *T is disconnected;*

(ii) *T has a partition into two non-empty open sets;*

(iii) *T has a partition into two non-empty closed sets;*

(iv) *T has a subset that is both open and closed and is neither $\varnothing$ nor T;*

(v) *there is a continuous function from T onto the two-point set $\{0, 1\}$ with the discrete topology.*

Proof. (i) $\Leftrightarrow$ (ii): This follows by definition of what it means to be 'not connected'
(ii) $\Leftrightarrow$ (iii): If $T = A \cup B$ with A and B open (closed) then $B = T \setminus A$ is closed (open) and $A = T \setminus B$ is closed (open).
(ii) $\Leftrightarrow$ (iv): Assume (ii), so we can write $T = A \cup B$, with A, B open, $A \cap B = \varnothing$ and $A, B \neq \varnothing, T$. As argued above, A and B are also closed, so (iv) holds. Now assume (iv) and let $A \subset T$ be both open and closed and neither $\varnothing$ or T. Then $B = T \setminus A$ is open and A, B give a partition of T.
(ii) $\Rightarrow$ (v): Assume (ii). Then we can write $T = A \cup B$, A, B open, with $A \cap B = \varnothing$. Set $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$. Then $f^{-1}(0) = A$, $f^{-1}(1) = B$, and $f^{-1}(\{0, 1\}) = T$, so the inverse image of all open sets are open and f is continuous.
(v) $\Rightarrow$ (ii): Assume that $f : T \to \{0, 1\}$ is a continuous surjection. Set $A = f^{-1}(0)$ and $B = f^{-1}(1)$; both of these sets are open since f is continuous; they are non-empty since f is onto; and $A \cup B = T$ and $A \cap B = \varnothing$. □

Note that we can use (v) to show that a space is connected by showing that any continuous function $f : T \to \{0, 1\}$ must be constant.

It follows from (iv) that if a space T is connected then if a subset A of T is both open and closed then it must be empty or all of T.

Definition 7.4. A subset S of T is connected/disconnected if $(S, \mathcal{T}_S)$ is connected/disconnected (i.e. S is connected/disconnected using the subspace topology).

In general to decide on connectedness of subsets we need the following definition.

Definition 7.5. A set $S \subset T$ is *separated* by subsets $U, V \in \mathcal{T}$ if

$$S \subset U \cup V, \qquad U \cap V \cap S = \varnothing, \qquad U \cap S \neq \varnothing, \qquad V \cap S \neq \varnothing.$$

Proposition 7.6. *A subspace S of a topological space T is disconnected if and only if it is separated by some open subsets $U, V \in \mathcal{T}$.*

Proof. If S is disconnected then there are non-empty $A, B \subset \mathcal{T}_S$ such that $S = A \cup B$ and $A \cap B = \varnothing$. By the definition of the subspace topology, there exist $U, V \in \mathcal{T}$ such that $A = U \cap S$ and $B = V \cap S$. Then U and V separate S.

Conversely, if U and V separate S then $U \cap S$, $V \cap S$ partition S. □

7.2 Connected subsets of $\mathbb{R}$

We will show that any connected subset of $\mathbb{R}$ must be an interval. By 'an interval' we mean a set of the form

- $\varnothing$, $\{a\}$ for any $a \in \mathbb{R}$;
- $[a, b]$ where $a < b$;
- $(a, b]$ where $a < b$ and $a = -\infty$ is allowed;
- $[a, b)$ where $a < b$ and $b = \infty$ is allowed;
- (a, b) where $a < b$ and $a = -\infty$ and $b = \infty$ are allowed.

Lemma 7.7. *A set $I \subset \mathbb{R}$ is an interval if and only if whenever $x, y \in I$ and $x < z < y$ we have $z \in I$.*

Proof. The intervals listed all have this property. We show the converse. Given $I \subset \mathbb{R}$ with this property, let $a = \inf I$ and $b = \sup I$. Certainly $(a, b) \subset I$: if $z \in (a, b)$ then there exists $\alpha, \beta \in I$ with $\alpha < z < \beta$ (by the definition of a and b), which implies that $z \in I$. Now

$$(a, b) \subset I \subset (a, b) \cup \{a, b\},$$

depending on whether $a, b \in I$. □

Theorem 7.8. *A subset of $\mathbb{R}$ is connected if and only if it is an interval.*

Proof. Step 1: If $I \subset \mathbb{R}$ is connected then it is an interval. Suppose that I is not an interval. Then there exist x, y, z such that $x < z < y$, $x, y \in I$ and $z \notin I$. Let $A = (-\infty, z) \cap I$ and $B = (z, \infty) \cap I$. Then A and B are disjoint, open in I (by definition of the subspace topology on I), and non-empty (since $x \in A$ and $y \in B$). We also have $I = A \cup B$, since $z \notin I$. So I is not connected, a contradiction.

[If you prefer you can think of this as a contrapositive argument: if I is not an interval then it is not connected; so any connected set is an interval.]

Step 2: Any interval is a connected set in $\mathbb{R}$. If I is not connected then there is a continuous surjective map $f : I \to \{0,1\}$. Note that if we consider $f : I \to \mathbb{R}$ then this is also continuous, since given any open subset U of $\mathbb{R}$ we have

$$f^{-1}(U) = \begin{cases} f^{-1}(\{0\}) & 0 \in U,\ 1 \notin U, \\ f^{-1}(\{1\}) & 1 \in U, 0 \notin U, \\ f^{-1}(\{0\}) \cup f^{-1}(\{1\}) & 0,1 \in U, \\ \varnothing & 0,1 \notin U, \end{cases}$$

and all these sets are open.

But if $f(x) = 0$ and $f(y) = 1$, the Intermediate Value Theorem implies that f takes all values in between, which is not possible. □

7.3 Operations on connected sets

Proposition 7.9. *Suppose that C_j, $j \in \mathcal{J}$, are connected subsets of T and $C_i \cap C_j \neq \varnothing$ for each i, j, then*

$$K = \bigcup_{j \in \mathcal{J}} C_j$$

is connected.

Proof. Suppose that $f : K \to \{0,1\}$ is continuous. Since each C_j is connected, $f(C_j) = \{\delta_j\}$, where $\delta_j = 0$ or 1 for each j. Since $C_i \cap C_j$ is always non-empty, it follows that $f(C_j)$ takes the same value for every $j \in \mathcal{J}$. So f cannot be onto and K is connected. □

Lemma 7.10. *Suppose that C_1 and C_2 are connected subsets of T and $\overline{C_1} \cap C_2 \neq \varnothing$. Then $C_1 \cup C_2$ is connected.*

Proof. Let $K = C_1 \cup C_2$ and suppose that $f : K \to \{0,1\}$ is continuous. Then $f(C_1) = \{0\}$, say. Suppose that $f(C_2) = \{1\}$. Then $f^{-1}(\{1\})$ is an open subset of K, so is given by $U \cap K$ for some open set U in T.

Now, since $\overline{C_1} \cap C_2$ is non-empty, there is a point $x \in C_2$ such that any open neighbourhood of x in T intersects C_1. The set U is one such set, so

$$U \cap C_1 \neq \varnothing.$$

But $C_1 \subset K$ so this is the same as

$$U \cap K \cap C_1 \neq \varnothing \qquad f^{-1}(\{1\}) \cap C_1 \neq \varnothing;$$

this is a contradiction (since $f(C_1) = \{0\}$), so f cannot be onto. □

Theorem 7.11. *Suppose that C and C_j ($j \in \mathcal{J}$) are connected subsets of T and $C_j \cap \overline{C} \neq \varnothing$ for each j. Then*

$$K = C \cup \bigcup_{j \in \mathcal{J}} C_j$$

is connected.

Proof. Set $C_i' = C \cup C_i$. Then each C_i' is connected (by Lemma 7.10), $C_i' \cap C_j' \neq \varnothing$ for every i, j, and $K = \bigcup_j C_j'$. The result now follows from Proposition 7.9. □

Corollary 7.12. *If $C \subset T$ is connected then so is any set K satisfying $C \subset K \subset \overline{C}$.*

Proof. We have $K = C \cup \bigcup_{x \in K} \{x\}$ and $\{x\} \cap \overline{C} \neq \varnothing$ for each $x \in K$. □

Theorem 7.13. *The continuous image of a connected set is connected.*

Proof. Suppose that $f : T \to S$ is continuous and that T is connected. If $f(T)$ is not connected then there exists a surjective continuous map $g : f(T) \to \{0, 1\}$. But then $g \circ f : T \to \{0, 1\}$ is continuous and surjective, contradicting the connectedness of T. □

This shows that connectedness is a topological property, i.e. if T and S are homeomorphic and T is connected then S is connected.

Theorem 7.14. *The product of two connected spaces is connected.*

Proof. Let T and S be two connected sets, and pick $s_0 \in S$. Define $C := T \times \{s_0\}$ and $C_t := \{t\} \times S$. Then C is homeomorphic to T and C_t is homeomorphic to S, so both are connected. We have $C_t \cap C \neq \varnothing$ (both sets contain (t, s_0)) and

$$T \times S = C \cup \bigcup_{t \in T} C_t,$$

so the result now follows from Theorem 7.11. □

Examples: To show that a set is connected, we construct it from continuous images of connected sets (e.g. intervals), via products or unions.

- $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is connected (Theorem 7.14).
- Circles are connected (the continuous image of intervals).
- $\mathbb{R}^2 \setminus \{0\}$ is connected (the union of circles about $(0, 0)$, each of which intersects the positive x axis).
- The 'topologist's sine curve'

  $$\mathscr{S} := \left\{ \left(x, \sin \frac{1}{x} \right) : \ x \in \mathbb{R}, \ x \neq 0 \right\} \cup \{(0, 0)\}$$

  is connected.

  Put $S_- = \{(x, \sin(1/x)) : x < 0\}$, $S_+ = \{(x, \sin(1/x)) : x > 0\}$, and $O = \{(0, 0)\}$; all three of these sets are connected. O is a point so connected, and S_- and S_+ are connected as images of the intervals $(-\infty, 0)$ and $(0, \infty)$ respectively under the continuous map $x \mapsto (x, \sin(1/x))$.

  Note that $O \subset \overline{S_-}$ and $O \subset \overline{S_+}$; so both $S_- \cup O$ and $S_+ \cup O$ are connected using Lemma 7.10. Now since $(S_- \cup O) \cap (S_+ \cup O)) = O \neq \varnothing$ it follows that

  $$\mathscr{S} = S_- \cup O \cup S_+$$

  is connected using Proposition 7.9.

- The 'harmonic comb'

$$\mathscr{H} = \{(x, y) : y = 0,\ x \in (0, 1]\} \cup \{(1/n, y) : n \in \mathbb{N},\ 0 \le y \le 1\} \cup \{(0, 1)\}$$

  is connected. It is the union of vertical lines, all of which intersect the horizontal line, plus $(0, 1)$, which is contained in the closure of the vertical lines.

We already observed that connectedness is a topological property. Often more useful in examples is the resulting fact that the property "$T \setminus \{x\}$ is connected for every $x \in T$" is a topological property: if $f : T \to S$ is a homeomorphism then for any $y \in S$ the set $S \setminus \{y\}$ is the continuous image of $T \setminus \{x\}$ for some $x \in X$.

We can use this to show that certain sets are not homeomorphic. For example:

- $[0, 1]$ is not homeomorphic to a circle: $[0, 1/2) \cup (1/2, 1]$ is disconnected, but with a point removed the circle is still connected.
- $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$: $(-\infty, 0) \cup (0, \infty)$ is disconnected, but $\mathbb{R}^2 \setminus \{0\}$ (and so $\mathbb{R}^2$ minus any point) is connected (as we observed above).
- $[0, 1]$ is not homeomorphic to a square: again, $[0, 1/2) \cup (1/2, 1]$ is disconnected, but the square minus a point is connected.

7.4 Equivalence relations

This subsection is not examinable in itself but helps understanding the next one.

Let X be any set (like the integers $\mathbb{Z}$). A *binary relation* R on X is simply a certain set of pairs (x, y) (where $x, y \in X$). For example, the pairs (n, m) for which n divides m; or the set of pairs (n, m) for which $n \le m$. The notation is either $(x, y) \in R$ or xRy.

An *equivalence relation* on a set X is a binary relation with special properties: it is *reflexive* (xRx for all x), *symmetric* (xRy implies yRx) and *transitive* (if xRy and yRz then xRz). For example, for $X = \mathbb{Z}$, consider the binary relation R such that nRm iff n and m have the same remainder when divided by 5. This is an equivalence relation. Equivalence relations are usually denoted by $\sim$, not the letter R.

For an equivalence relation $\sim$, we can form the equivalence class of an arbitrary $x \in X$,

$$[x] = \{y \in X : x \sim y\},$$

this is the set of all those elements of X that are "equivalent" to x. That is, if $\sim$ is the equivalence relation on $\mathbb{Z}$ for which $n \sim m$ iff n and m have the same remainder when divided by 5, then $[3]$ is the set of all integers whose remainder is 3 when divided by 5. Also, $[0] = [5]$. In this example, there are exactly 5 equivalence classes.

7.5 Connected components

We can define an *equivalence relation* on a topological space T by letting $x \sim y$ if there is a connected set $C \subset T$ such that $x, y \in C$.

This is clearly reflexive and symmetric; transitivity comes from Theorem 7.11: if $x \sim y$ and $y \sim z$ then $x, y \in C_1$, $y, z \in C_2$ and $C_1 \cap C_2 \neq \varnothing$, so $C_1 \cup C_2$ is connected and $x, z \in C_1 \cup C_2$.

Definition 7.15. The equivalence classes of $\sim$ are called the *connected components of T*.

We have the following:

- the connected component containing x is the union of all connected subsets of T that contain x;
- connected components are connected (Theorem 7.11);
- connected components are closed (Corollary 7.12);
- connected components are maximal connected subsets of T (i.e. if C is a connected component and $C \subset D$ with D connected then $C = D$.

Examples: the connected components of $(0,1)\cup(1,2)$ are $(0,1)$ and $(1,2)$; the connected components of $\mathbb{Q}$, $\mathbb{R}\setminus\mathbb{Q}$, the Cantor set are all points.

Since the continuous image of a connected space is connected, the number of connected components is a topological property.

7.6 Path-connected spaces

Definition 7.16. If $u, v \in T$ a *path* from u to v is a continuous map $\varphi : [0,1] \to T$ such that $\varphi(0) = u$ and $\varphi(1) = v$. A space T is path connected if any two points in T can be joined by a path in T.

Proposition 7.17. *A path-connected space T is connected.*

Proof. Fix $u \in T$, and consider any $v \in T$. Then there is a path from u to v, and so C_v, the image of the map φ is connected (it is the continuous image of $[0,1]$). Then $T = \{u\} \cup \bigcup_{v\in T} C_v$, and each C_v contains u, so T is connected using Theorem 7.11. □

The converse is in general not true, e.g. the 'hamonic comb' is not path connected (and nor is the topologist's sine curve, but this is harder to show).

Not examinable

7.7 Open sets in $\mathbb{R}^n$

In some 'nice' situations connected sets are path connected.

Theorem 7.18. *Connected open subsets of $\mathbb{R}^n$ are path connected.*

Before the proof we make two observations.

1. If we have two paths, φ_1 from a to b and φ_2 from b to c, then we combine them to give a path from a to c by setting

$$\varphi(t) := \begin{cases} \varphi_1(2t), & 0 \le t \le 1/2, \\ \varphi_2(2t-1), & 1/2 < t \le 1; \end{cases}$$

the function φ is continuous since $\varphi_1(1) = \varphi_2(0)$.

2. Any open ball in $\mathbb{R}^n$ is path connected. Take the ball $B(a,\epsilon)$, and $x, y \in B(a,\epsilon)$. Then either consider the path that joins x to a and then goes from a to y:

$$\varphi(t) := \begin{cases} x + 2t(a-x) & 0 \le t < 1/2 \\ a + (2t-1)(y-a) & 1/2 \le t \le 1, \end{cases}$$

or move along a straight line from x to y on the path

$$\varphi(t) = (1-t)x + ty.$$

It is easy to see that the first path lies entirely in $B(a,\epsilon)$; for the second it follows from the fact that $B(a,\epsilon)$ is convex, which we can prove easily by observing that

$$|[(1-t)x + ty] - a| = |[(1-t)(x-a) + t(y-a)| \le (1-t)|x-a| + t|y-a| < \epsilon.$$

Proof. Let U be a connected open subset in $\mathbb{R}^n$.

Take $u \in U$ and let A be the set of all points in U that can be reached by a path in U. Let $B = U \setminus A$; we will show that B is empty by proving that if it is not that A and B form a partition of U.

First we show that A is open. Take any $a \in A$; since U is open we have $B(a,\epsilon) \subset U$ for some $\epsilon > 0$. So there is a path joining a to any $x \in B(a,\epsilon)$. By combining the path from u to a with this path from a to x we obtain a path from u to x, so $B(a,\epsilon) \subset A$, i.e. A is open.

The set B is also open: for any $y \in B$ we have $B(y,\epsilon) \subset U$; if there was a path from u to $z \in B(y,\epsilon)$ there would be a path from u to y, so we must have $B(y,\epsilon) \subset B$.

Now if B is non-empty we have $U = A \cup B$, $A \cap B \cap U = \varnothing$, $A \cap U \ne \varnothing$, $B \cap U \ne \varnothing$. But U is connected. □

Theorem 7.19. *Open subsets of $\mathbb{R}^n$ have open connected components.*

Proof. Let U be an open subset of $\mathbb{R}^n$ and C one of its connected components. If $x \in C$ then there exists $\delta > 0$ such that $B(x,\delta) \subset U$. But C is the union of all connected subsets of U that contain x, so $B(x,\delta) \subset C$, so C is open. □

Theorem 7.20. *A subset U of $\mathbb{R}$ is open if and only if it is the disjoint union of countably many open intervals, i.e. $U = \bigcup_{j \in \mathcal{J}} (a_j, b_j)$, with the intervals disjoint and $\mathcal{J}$ finite or countably infinite.*

Proof. Any union of open intervals is open, so we need to prove that any open set can be written in this form.

Take any open $U \subset \mathbb{R}$ and let $\{U_j\}_{j \in \mathcal{J}}$ be the collection of all its connected components, which are mutually disjoint. We have just shown that they are open; since they are open and connected, they are open intervals. For each C_j we can choose a rational $q_j \in C_j$. Since the rationals are countable, so are the C_j. □

8 Completeness in metric spaces

8.1 Completeness

Recall that if a sequence (x_n) converges in a metric space (X,d) then it is Cauchy, i.e. for every $\epsilon > 0$ there exists N such that

$$d(x_n, x_m) < \epsilon \qquad \text{for every } n, m \geq N$$

(see remarks after the statement of Theorem 6.28). A theorem from Analysis guarantees that any Cauchy sequence in $\mathbb{R}$ converges.

Definition 8.1. A metric space (X,d) is *complete* if any Cauchy sequence in X converges.

It is implicit in the definition that the limit of the sequence must lie in X. So $\mathbb{R}$ is complete, $\mathbb{C}$ is complete, but $(0,1)$ is not complete (the sequence $x_n = 1 - 1/n$ converges, but its limit 1 does not lie in $(0,1)$). Since $\mathbb{R}$ and $(0,1)$ are homeomorphic, this shows that *completeness is not a topological property.*

Recall that a subset K of a metric space (X,d) is closed if whenever $(x_n) \in K$ and $x_n \to x$, then $x \in K$ (Lemma 3.23).

Proposition 8.2. *Suppose that (X,d) is a metric space and that S is a subset of X. If $(S, d|_S)$ is complete then S is a closed subset of X, and if (X,d) is complete and S is closed then $(S, d|_S)$ is complete.*

Proof. Suppose that $(x_n) \in S$ with $x_n \to x$. Then (x_n) is Cauchy in S, so converges to some $y \in S$. Since $x_n, y \in S$, it follows that $d|_S(x_n, y) = d(x_n, y)$, so $x_n \to y$ in X, i.e. $x = y$ and S is closed.

If (x_n) is Cauchy in S then (x_n) is also Cauchy in X, so converges to some $x \in X$; since S is closed, $x \in S$, so $x_n \to x$ in S. □

Proposition 8.3. *Any compact metric space (X,d) is complete.*

Proof. If (x_n) is a Cauchy sequence in X then it has a convergent subsequence (since compact implies sequentially compact in a metric space) with $x_{n_j} \to x \in X$. But if a Cauchy sequence has a convergent subsequence then the whole sequence converges to x: given any $\epsilon > 0$, find N such that

$$d(x_n, x_m) < \epsilon/2 \qquad n, m \geq N$$

and J such that $n_J \geq N$ and $d(x_{n_j}, x) < \epsilon/2$ for all $j \geq J$. Then for all $k \geq n_J$ we have

$$d(x_k, x) \leq d(x_k, x_{n_J}) + d(x_{n_J}, x) < \epsilon/2 + \epsilon/2 = \epsilon.$$ □

8.2 Examples of complete spaces

Note that all our examples will be normed spaces. A normed space is complete if it is complete as a metric space, i.e. a Cauchy sequence is (x_n) such that for every $\epsilon > 0$ there exists N such that

$$\|x_n - x_m\| < \epsilon \qquad n, m \geq N,$$

and any such sequence should converge to some $x \in X$, i.e. $\|x_n - x\| \to 0$ as $n \to \infty$.

Theorem 8.4. *$\mathbb{R}^d$ is complete.*

Proof. Let $(x^{(k)})_{k=1}^\infty$ be a Cauchy sequence in $\mathbb{R}^d$. Then for every $\epsilon > 0$ there exists $N(\epsilon)$ such that

$$\|x^{(n)} - x^{(m)}\| = \left(\sum_{i=1}^d |x_i^{(n)} - x_i^{(m)}|^2 \right)^{1/2} < \epsilon \qquad \text{for } m, n \geq N(\epsilon).$$

In particular, for each $i = 1, \ldots, d$ we have

$$|x_i^{(n)} - x_i^{(m)}| < \epsilon \qquad \text{for } m, n \geq N(\epsilon),$$

so $(x_i^{(n)})_{n=1}^\infty$ is a Cauchy sequence. Since Cauchy sequences of real numbers converge, $x_i^{(n)} \to x_i$ for some $x_i \in \mathbb{R}$.

Now set $x = (x_1, \ldots, x_d)$; then

$$\lim_{n\to\infty} \|x^{(n)} - x\| = \lim_{n\to\infty} \left(\sum_{i=1}^d |x_i^{(n)} - x_i|^2 \right)^{1/2} = 0,$$

so $x^{(n)} \to x$. □

The above theorem referred to the standard norm but, since all norms on $\mathbb{R}^d$ are equivalent, $\mathbb{R}^d$ is complete in any norm.

Theorem 8.5. *For every $1 \leq p \leq \infty$, ℓ^p is complete.*

Proof. This is an exercise on Problem Sheet 9. □

Theorem 8.6. *For any non-empty set X, the space $B(X)$ of bounded real-valued functions on X, $f : X \to \mathbb{R}$, with the 'sup norm'*

$$\|f\|_\infty := \sup_{x\in X} |f(x)|$$

is complete.

Not examinable

Proof. Let (f_n) be a Cauchy sequence in $B(X)$. Then for every $\epsilon > 0$ there exists $N(\epsilon)$ such that

$$\|f_n - f_m\|_\infty = \sup_{x\in X} |f_n(x) - f_m(x)| < \epsilon \qquad \text{for } n, m \geq N(\epsilon).$$

In particular, for each $x \in X$ we have

$$|f_n(x) - f_m(x)| < \epsilon \qquad \text{for } n, m \geq N(\epsilon), \tag{8}$$

so $(f_n(x))_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $f_n(x)$ converges for each $x \in X$.

Now we define $f : X \to \mathbb{R}$ by setting

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for each $x \in X$. For any $\epsilon > 0$ we have

$$|f_n(x) - f(x)| \le \epsilon \qquad \text{for } n \ge N(\epsilon),$$

letting $m \to \infty$ in equation (8).

Since $N(\epsilon)$ does not depend on x this implies (i) that

$$|f_{N(1)}(x) - f(x)| \le 1 \qquad \text{for every } x \in X,$$

so f is bounded, i.e. an element of $B(X)$ and (ii) that

$$\|f_n - f\|_\infty \le \epsilon \qquad \text{for all } n \ge N(\epsilon),$$

i.e. that $f_n \to f$ in the sup metric. □

Theorem 8.7. *The space $C_b(T)$ of all bounded continuous functions from any non-empty topological space T into $\mathbb{R}$ ('continuous real-valued functions') is a closed subspace of $B(T)$, and hence complete.*

Not examinable

Proof. Suppose that $f \in \overline{C_b(T)}$, where the closure is taken in $B(T)$. Then for any $\epsilon > 0$ there exists $f_\epsilon \in C_b(T)$ such that $\|f - f_\epsilon\|_\infty < \epsilon$.

Now, we will show that for any $a \in \mathbb{R}$ we have

$$\{x : f(x) > a\} = \bigcup_{\epsilon > 0} \{x : f_\epsilon(x) > a + \epsilon\}. \qquad (*)$$

Indeed, if $f(x) > a$ then we can take $\epsilon = (f(x) - a)/2$ and then

$$f_\epsilon(x) = f(x) - (f(x) - f_\epsilon(x)) > f(x) - \epsilon = a + \epsilon;$$

while if $f_\epsilon(x) > a + \epsilon$ then

$$f(x) = f_\epsilon(x) - (f_\epsilon(x) - f(x)) > (a + \epsilon) - \epsilon = a.$$

Now we have $(*)$, we note that, since each f_ϵ is continuous, each set in the union on the right-hand side is open, and so $f^{-1}(a, \infty)$ is open.

A similar argument works for $\{x : f(x) < a\}$, and continuity of f now follows from Lemma 5.29, since

$$\{(a, \infty), (-\infty, a) : a \in \mathbb{R}\}$$

forms a sub-basis for the open sets of $\mathbb{R}$. □

Simpler argument if T is a metric space. Suppose that $f_n \in C_b(T)$ and $f_n \to f$ in the sup norm.

Take $x \in X$. We show that f is continuous at x. Fix $\epsilon > 0$, and find N such that

$$\|f_n - f\|_\infty < \epsilon/3 \qquad n \ge N.$$

Since f_N is continuous at x there exists $\delta > 0$ such that $d(y,x) < \delta$ implies that

$$|f_N(x) - f_N(y)| < \epsilon/3.$$

It follows that if $d(y,x) < \delta$ then

$$\begin{aligned}|f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq \|f - f_n\|_\infty + \frac{\epsilon}{3} + \|f_n - f\|_\infty \\ &= \epsilon.\end{aligned}$$

□

Corollary 8.8. *If T is non-empty and compact then $C(T)$ is complete with the maximum norm*

$$\|f\|_\infty = \max_{x\in T} |f(x)|.$$

Proof. If $f \in C(T)$ and T is compact then f is bounded, so $C(T) = C_b(T)$, and f attains its bounds, so $\sup_{x\in T} |f(x)| = \max_{x\in T} |f(x)|$. □

Note that in fact these are all normed spaces. A complete normed space is called a Banach space.

8.3 Completions

Consider the space $C[0,1]$ with the L^1 norm. We can find a sequence that is Cauchy in the L^1 norm but that does not converge to a function in $C[0,1]$. Consider the sequence for $n \geq 2$ given by

$$f_n(x) = \begin{cases} 0 & 0 \leq x < 1/2 - 1/n \\ 1 - n(1/2 - x) & 1/2 - 1/n \leq x \leq 1/2 \\ 1 & 1/2 < x \leq 1. \end{cases}$$

Then for $n > m$ we have

$$\|f_n - f_m\|_{L^1} = \int_0^1 |f_n(x) - f_m(x)|\, dx \leq \frac{1}{m};$$

so this sequence is Cauchy.

It converges in the L^1 norm to the function

$$f(x) = \begin{cases} 0 & 0 \leq x < 1/2 \\ 1 & 1/2 \leq x \leq 1, \end{cases}$$

since

$$\|f_n - f\|_{L^1} = \int_0^1 |f_n(x) - f(x)|\, dx = \int_{1/2-1/n}^{1/2} |f_n(x)|\, dx \leq \frac{1}{n}.$$

Clearly $f \notin C[0,1]$. With some additional arguments, this implies that $C[0,1]$ with the L^1 norm is not complete.

There is a way of 'completing' a space A by 'adding the missing limits'. There are two ways of doing this. First recall that a subset A of metric space is *dense* in X if $\overline{A} = X$.

Method 1: Find a complete metric space X that contains A such that $\overline{A} = X$. E.g. $\mathbb{R}$ is the completion of $\mathbb{Q}$.

Method 2: Find a complete metric space X and an isometry $\mathfrak{i} : A \to Y$ with $Y \subset X$ and $\overline{Y} = X$. $\mathbb{R}$ is the completion of $\mathbb{Q}$ with $\mathfrak{i}(x) = x$ or with $\mathfrak{i}(x) = -x$.

The advantage of the second method is that we can 'construct' X rather than having to find a space X that already contains A. One can show that completions are unique, in the sense given any two completions $(X, \mathfrak{i})$ and $(X', \mathfrak{i}')$ there is an isometry $\mathfrak{j} : X \to X'$ with $\mathfrak{j}(\mathfrak{i}(x)) = \mathfrak{i}'(x)$.

One can make an abstract 'completion' by using the following result.

Theorem 8.9. *Any metric space (X, d) can be isometrically embedded into the complete metric space $B(X)$.*

Proof. Given (X, d), define $\mathfrak{i} : X \to B(X)$ by choosing some $a \in X$ and then setting

$$[\mathfrak{i}(x)](z) = d(z, x) - d(z, a).$$

Note that for every $z \in X$ we have

$$|[\mathfrak{i}(x)](z)| = |d(z, x) - d(z, a)| \leq d(x, a),$$

so $\mathfrak{i}(x) \in B(X)$. Since

$$|[\mathfrak{i}(x)](z) - [\mathfrak{i}(y)](z)| = |d(z, x) - d(z, y)| \leq d(x, y)$$

and we have equality when $z = x$ or $z = y$, it follows that

$$\|\mathfrak{i}(x) - \mathfrak{i}(y)\|_\infty = d(x, y),$$

so the map $\mathfrak{i}$ is an isometry of (X, d) onto a subset of $B(X)$. □

Corollary 8.10. *Any metric space has a completion.*

Proof. Embed (X, d) into $B(X)$ using Theorem 8.9. Then $\overline{\mathfrak{i}(X)}$ (with the closure taken in $B(X)$) is a closed subset of a complete space, so complete by Proposition 8.2. Clearly $\mathfrak{i}(X)$ is dense in $\overline{\mathfrak{i}(X)}$. □

One can also complete any normed space to find a complete normed space; but the construction is significantly more involved.

8.4 The Contraction Mapping Theorem

Definition 8.11. A map $f : X \to X$ is a *contraction* if

$$d(f(x), f(y)) \leq \kappa\, d(x, y), \qquad x, y \in X,$$

for some $\kappa < 1$.

Any contraction is continuous: so if $x_n \to x$ we have $f(x_n) \to f(x)$ (we will use this in the proof of the following theorem).

Theorem 8.12 (Contraction Mapping Theorem)**.** *Let (X,d) be a non-empty complete metric space and $f : X \to X$ a contraction. Then f has a unique fixed point in X, i.e. there exists a unique $x \in X$ such that $f(x) = x$.*

This is also known as Banach's Fixed Point Theorem.

Proof. Choose any $x_0 \in X$ and set $x_{n+1} = f(x_n)$. Then

$$d(x_{j+1}, x_j) \le \kappa d(x_j, x_{j-1}) \le \kappa^2 d(x_{j-1}, x_{j-2}) \le \cdots \le \kappa^j d(x_1, x_0),$$

so if $k > j$

$$d(x_k, x_j) \le \sum_{i=j}^{k-1} d(x_{i+1}, x_i) \le \sum_{i=j}^{k-1} \kappa^i d(x_1, x_0) \le \frac{\kappa^j}{1-\kappa} d(x_1, x_0).$$

It follows that (x_n) is a Cauchy sequence in X. Since X is complete, $x_n \to x$ for some $x \in X$. Since f is continuous we have $f(x_n) \to f(x)$. Now take limits on both sides of

$$x_{n+1} = f(x_n)$$

to show that $x = f(x)$.

Any such x must be unique, since if $f(x) = x$ and $f(y) = y$ it follows that

$$d(x,y) = d(f(x), f(y)) \le \kappa d(x,y) \qquad \Rightarrow \qquad (1-\kappa) d(x,y) = 0,$$

so $x = y$. □

As an application we can prove the local existence and uniqueness of solutions of ordinary differential equations.

Theorem 8.13 (Picard–Lindelöf Theorem)**.** *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous with*

$$|f(x) - f(y)| \le L|x - y| \qquad x, y \in \mathbb{R}^n.$$

Then for any $x_0 \in \mathbb{R}^n$ the differential equation

$$\dot{x} = f(x) \qquad x(0) = x_0$$

has a unique solution on $[-T, T]$ for any $LT < 1$.

Proof. Rewrite the equation in the form

$$x(t) = x_0 + \int_0^t f(x(s))\, ds.$$

So $x : [-T, T] \to \mathbb{R}^n$ solves the ODE if it is a fixed point of the map

$$\mathcal{F} : C([-T, T]) \to C([-T, T])$$

given by

$$[\mathcal{F}(x)](t) := x_0 + \int_0^t f(x(s))\,ds.$$

We use the Contraction Mapping Theorem in the space $X := C([-T,T])$ with the supremum metric.

This map $\mathcal{F}$ is a contraction on X if $LT < 1$, since

$$\begin{aligned}
|[\mathcal{F}(x)](t) - [\mathcal{F}(y)](t)| &= \left|\int_0^t f(x(s)) - f(y(s))\,ds\right| \\
&\le \int_0^t |f(x(s)) - f(y(s))|\,ds \\
&\le \int_0^t L|x(s) - y(s)|\,ds \\
&\le LT\|x - y\|_\infty,
\end{aligned}$$

so

$$\|\mathcal{F}(x) - \mathcal{F}(y)\|_\infty \le LT\,\|x - y\|_\infty. \qquad \square$$

(In this proof, by $C([-T,T])$ we meant the space of continuous functions $x : [-T,T] \to \mathbb{R}^n$ in the maximum norm

$$\|x\|_\infty = \max_{t\in[-T,T]} |x(t)|.$$

This is a complete metric space for every n.)

8.5 The Arzelà–Ascoli Theorem

Definition. Let X be a metric space. A family A of continuous functions $X \to \mathbb{R}$ is

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- *equicontinuous at x* if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d(x,y) < \delta \qquad \Rightarrow \qquad |f(y) - f(x)| < \epsilon \qquad \text{for every } f \in A;$$

- *equicontinuous* if it is equicontinuous at every $x \in X$;

Definition 8.14. Let X be a metric space. A family A of continuous functions $X \to \mathbb{R}$ is

- *uniformly equicontinuous* if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d(y,x) < \delta \qquad \Rightarrow \qquad |f(y) - f(x)| < \epsilon \qquad \text{for every } f \in A.$$

Remark: If X is compact then $A \subset C(X)$ is equicontinuous if and only if it is uniformly equicontinuous; see Problem Sheet 9.

Definition 8.15. We say that a sequence of functions (f_n) is uniformly bounded if there is a real number M such that $\|f_n\|_\infty \le M$ for all n.

Theorem 8.16 (Arzelà–Ascoli Theorem). *Let X be a compact metric space. Suppose that the sequence (f_n) in $C(X)$ is uniformly bounded and uniformly equicontinuous. Then (f_n) has a subsequence that converges in the maximum norm to a function $f \in C(X)$.*

Before we prove this theorem, here is an application.

Theorem 8.17 (Peano). *Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous. Then there exists $T > 0$ such that the differential equation*

$$\dot{x}(t) = f(t, x(t)), \qquad x(0) = x_0$$

has at least one solution for $t \in (-T, T)$.

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Structure of the proof. Assume, for simplicity, that $x_0 = 0$. First we construct "approximate solutions". For each positive integer n, let $x_n : [0, \infty) \to \mathbb{R}$ be the unique continuous function that is linear on each of the intervals $[i/n, (i+1)/n)$ such that the (right) derivatives satisfy

$$\dot{x}_n(t) = f\Big(i/n,\ x_n(i/n)\Big) \quad \text{if } t \in \left[\frac{i}{n}, \frac{i+1}{n}\right)$$

for every integer $i \geq 0$; and $x(0) = x_0 = 0$.

Assume that $|f(t, x)| \leq M$ if $|t| \leq 1$ and $|x| \leq 1$. Set $T = \min(1, 1/M)$. Then the approximate solutions x_n on $[0, T]$ are uniformly bounded (by 1) and uniformly equicontinuous (for every $\epsilon > 0$, $\delta = \epsilon/M$ works).

By the Arzelà–Ascoli Theorem, (x_n) in $C([0, T])$ has a subsequence that converges in the maximum norm. One then shows that the limit is a solution of the differential equation for $t \in [0, T)$. □

Everything after this point is not examinable, as indicated.

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We can split the proof of Theorem 8.16 into three parts, each covered in a separate lemma below.

Lemma 8.18 (The diagonal subsequence). *Let $f_n : X \to \mathbb{R}$ be a uniformly bounded sequence of functions. Let $D = \{x_k : k = 1, 2, \ldots\}$ be a countable subset of X. Then (f_n) has a subsequence (f_{n_j}) such that the sequence of real numbers $f_{n_j}(x_k)$ converges (as $j \to \infty$) for each $x_k \in D$.*

Lemma 8.19. *Every compact metric space contains a countable dense set $D = \{x_k : k \in \mathbb{N}\}$.*

Lemma 8.20. *Let (X, d) be a compact metric space and let D be a dense subset of X. Let (f_n) be a uniformly equicontinuous sequence in $C(X)$ such that $f_n(x)$ converges for every $x \in D$. Then (f_n) converges in the maximum norm.*

Proof of Theorem 8.16. Since X is compact, there is a countable dense set $D \subset X$ by Lemma 8.19. Since (f_n) is uniformly bounded, we can apply Lemma 8.18 to obtain a subsequence $(f_{r_j})_{j=1}^{\infty}$ such that $f_{r_j}(x)$ converges for every $x \in D$. Since (f_n) and thus (f_{r_j}) are uniformly equicontinuous we can apply Lemma 8.20 to conclude that (f_{r_j}) converges in the maximum norm. □

Proof of Lemma 8.18. Note that since (f_n) is uniformly bounded, the sequence of real numbers $(f_n(x))$ is bounded for every $x \in X$.

Since $(f_n(x_1))$ is bounded, by the Bolzano–Weierstrass theorem (f_n) has a subsequence $(f_{n_{1,j}})$ such that $(f_{n_{1,j}}(x_1))$ converges. Let S_1 be the set of these indices, that is, $S_1 = \{n_{1,j} : j = 1, 2, \ldots\} \subset \mathbb{N}$.

Since $(f_{n_{1,j}}(x_2))$ is bounded (here $n_{1,j} \in S_1$), by Bolzano–Weierstrass $(f_{n_{1,j}})$ has a subsequence $(f_{n_{2,j}})$ such that $(f_{n_{2,j}}(x_2))$ converges. Let $S_2 = \{n_{2,j} : j = 1, 2, \ldots\}$, this is a subset of S_1.

We continue this way. Suppose $k-1$ steps have been completed and we already have a sequence $(f_{n_{k-1,j}})$ with the infinite set $S_{k-1} = \{n_{k-1,j} : j = 1, 2, \ldots\}$. Since $(f_{n_{k-1,j}}(x_k))$ is bounded, by Bolzano–Weierstrass $(f_{n_{k-1,j}})$ has a subsequence $(f_{n_{k,j}})$ such that $(f_{n_{k,j}}(x_k))$ converges. Let $S_k = \{n_{k,j} : j = 1, 2, \ldots\}$. Then

$$S_k \subset S_{k-1} \subset \cdots \subset S_1 \subset \mathbb{N}.$$

This process can be continued forever, for every $k \geq 1$.

We now select the "diagonal subsequence". For each positive integer j, let $r_j = n_{j,j}$, this is the j^{th} smallest number of S_j. Notice that for each k, at most the first $k-1$ terms of the sequence $(f_{r_j})_{j=1}^{\infty}$ are not included in the sequence $(f_{n_{k,j}})_{j=1}^{\infty}$ since

$$r_j = n_{j,j} \in S_j \subset S_k \quad \text{if} \quad j \geq k.$$

Therefore, as $f_{n_{k,j}}(x_k)$ converges, $f_{r_j}(x_k)$ converges too, for every $k \geq 1$. □

Proof of Lemma 8.19. For each positive integer n the open balls of radius $1/n$, $\{B(x, 1/n) : x \in X\}$, form an open cover of X. It has a finite subcover consisting of M_n balls

$$B(x_{n,1}, 1/n), \ldots, B(x_{n,M_n}, 1/n).$$

Let D be the countable set that contains all these points $x_{n,j}$ $(n \geq 1,\ 1 \leq j \leq M_n)$.

To show that D is dense, it is enough to prove that D intersects every open ball $B(y, r)$, where $y \in X$ and $r > 0$. Let n be such that $r > 1/n$. The point y must be an element of one of the balls $B(x_{n,j}, 1/n)$, therefore $d(y, x_{n,j}) < 1/n < r$. Then $x_{n,j} \in B(y, r) \cap D$, so $B(y, r) \cap D \neq \varnothing$. □

Proof of Lemma 8.20. Let $\epsilon > 0$. By uniform equicontinuity, there is a $\delta > 0$ such that $d(x, y) < \delta$ implies that $|f_n(x) - f_n(y)| < \epsilon$ for all n.

The family of open balls of radius $\delta/2$, $\{B(y, \delta/2) : y \in X\}$ is an open cover of the compact space X. Therefore there exist finitely many such balls $B(y_1, \delta/2), \ldots, B(y_M, \delta/2)$ that cover X.

Since D is dense in X there are points $x_i \in D \cap B(y_i, \delta/2)$ for $1 \leq i \leq M$. Since $x_i \in D$, $\lim_{n\to\infty} f_n(x_i)$ exists, hence there is an integer N_i such that

$$|f_m(x_i) - f_n(x_i)| < \epsilon \quad \text{if } n, m \geq N_i.$$

Let $N = \max_{1 \leq i \leq M} N_i$.

Let $x \in X$. Then $x \in B(y_i, \delta/2)$ for some i and $d(x, x_i) \leq d(x, y_i) + d(y_i, x_i) < \delta$. By the triangle inequality, if $m, n \geq N$ then we have

$$|f_m(x) - f_n(x)| \leq \underbrace{|f_m(x) - f_m(x_i)|}_{<\epsilon} + \underbrace{|f_m(x_i) - f_n(x_i)|}_{<\epsilon} + \underbrace{|f_n(x_i) - f_n(x)|}_{<\epsilon} < 3\epsilon$$

where we used uniform continuity and the choice of δ for the first and third summands, and the choice of N for the second summand. Taking maximum over all $x \in X$, we obtain that

$$\|f_m - f_n\|_\infty = \max_{x \in X} |f_m(x) - f_n(x)| \leq 3\epsilon \quad \text{if } m, n \geq N.$$

This means that (f_n) is a Cauchy sequence in the maximum norm (because $\epsilon > 0$ was arbitrary and N depends on ϵ). Since $C(X)$ is complete, (f_n) converges in the maximum norm. □

The Arzelà–Ascoli Theorem can be used to characterise compact subsets of $C(X)$.

Theorem 8.21 (Arzelà–Ascoli Theorem, general form)**.** *Let X be a compact metric space. A subset A of $C(X)$ is compact if and only if it is closed, bounded, and equicontinuous.*

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8.6 The Baire Category Theorem

If S is a non-empty subset of a metric space (X, d) we define

$$\operatorname{diam}(S) = \sup_{x,y \in S} d(x, y).$$

Note that S is bounded if and only if $\operatorname{diam}(S) < \infty$.

Theorem 8.22 (Cantor's Theorem)**.** *If (X, d) is complete metric space and (F_n) a decreasing sequence of non-empty closed subsets of X such that $\operatorname{diam}(F_n) \to 0$ then*

$$\bigcap_{n=1}^{\infty} F_n \neq \varnothing.$$

Proof. For each $n \in \mathbb{N}$ choose some $x_n \in F_n$. Then for all $i \geq n$ we have $x_i \in F_n$. So if $i, j \geq n$ we have $x_i, x_j \in F_n$, so $d(x_i, x_j) \leq \operatorname{diam}(F_n)$. It follows that (x_n) is Cauchy, and so $x_n \to x$ for some $x \in X$.

Since each F_n is closed and $x_i \in F_n$ for all $i \geq n$, it follows that $x \in F_n$ for each n. So $x \in \cap_{n=1}^{\infty} F_n$, i.e. the intersection is non-empty. □

Recall that $A \subset (X, d)$ is dense in X if $\overline{A} = X$.

Theorem 8.23. *Let (X,d) be a complete metric space and $\{G_k\}_{k=1}^{\infty}$ a countable collection of open dense subsets of X. Then*

$$G := \bigcap_{k=1}^{\infty} G_k$$

is dense in X.

A set is called *residual* if it contains a countable intersection of open dense sets (like G in the above theorem).

Proof. Take $x \in X$ and $r > 0$; we need to show that $B(x,r) \cap G$ is non-empty.

Since each G_n is open and dense, we can find $y \in G_n$ and $s > 0$ such that

$$B(x,r) \cap G_n \supset B(y,2s) \supset \overline{B(y,s)}.$$

First choose $x_1 \in X$ and $r_1 < 1/2$ such that

$$\overline{B(x_1,r_1)} \subset B(x,r) \cap G_1;$$

then take $x_2 \in X$ and $r_2 < 2^{-2}$ such that

$$\overline{B(x_2,r_2)} \subset B(x_1,r_1) \cap G_2;$$

and inductively $x_n \in X$ and $r_n < 2^{-n}$ such that

$$\overline{B(x_n,r_n)} \subset B(x_{n-1},r_{n-1}) \cap G_n.$$

This yields a sequence of nested closed sets,

$$\overline{B(x_1,r_1)} \supset \overline{B(x_2,r_2)} \supset \overline{B(x_3,r_3)} \supset \cdots . \tag{9}$$

Since (X,d) is complete, by Cantor's theorem there exists $x_0 \in X$ such that

$$x_0 \in \bigcap_{j=1}^{\infty} \overline{B(x_j,r_j)}.$$

Now observe that $x_0 \in \overline{B(x_1,r_1)} \subset B_r(x)$, and that $x_0 \in \overline{B(x_n,r_n)} \subset G_n$ for every $n \in \mathbb{N}$. It follows that $x_0 \in B(x,r) \cap G$, and hence $B(x,r) \cap G$ is non-empty as claimed and G is dense in X. □

An alternative formulation says that you cannot make a complete metric space from the countable union of 'small' sets. Recall that a subset W of (X,d) is *nowhere dense* if $\overline{W}^{\circ} = \varnothing$. We showed previously that if W is nowhere dense then $X \setminus \overline{W}$ is open and dense (see comments after Definition 5.23): indeed, using Lemma 5.20 if W is nowhere dense then we have

$$\varnothing = (\overline{W})^\circ = X \setminus \overline{(X \setminus \overline{W})}$$

so

$$X = \overline{X \setminus \overline{W}}.$$

Corollary 8.24. *Let $\{F_j\}_{j=1}^\infty$ be a countable collection of nowhere dense subsets of a non-empty complete metric space (X,d). Then*

$$\bigcup_{j=1}^\infty F_j \neq X.$$

[“A complete metric space is not meagre in itself.”]

A countable union of nowhere dense subsets is called *meagre*.

Proof. The sets $X \setminus \bar{F}_j$ are a countable collection of open dense sets. It follows that

$$\bigcap_{j=1}^\infty X \setminus \bar{F}_j = X \setminus \bigcup_{j=1}^\infty \bar{F}_j$$

is dense, and in particular non-empty. □

Lemma 8.25. *The Cantor set is uncountable.*

Proof. Since C is a closed subset of $\mathbb{R}$, it is complete as a metric space. For every $x \in C$ there are points in C arbitrarily close to x. so $C \setminus \{x\}$ is dense in C. Since $\{x\}$ is closed this shows that $\{x\}$ is nowhere dense. Then we cannot have $C = \bigcup_{j=1}^\infty x_j$, so C is uncountable. □

(A similar proof can be used to show directly that $[0,1]$ is uncountable; although this follows from the above result since $C \subset [0,1]$.)

Not examinable

9 Appendices

Everything in the appendix is not examinable.

9.1 The topology of pointwise convergence

This appendix gives an example of a topological space which is Hausdorff but is *not* metrisable. **It is not examinable.**

Let $\mathcal{F}(X)$ denote the collection of all real-valued functions on X (i.e. all maps $f : X \to \mathbb{R}$).

Definition 9.1. The *topology of pointwise convergence* on $\mathcal{F}(X)$ is the topology $\mathcal{T}_p$ with sub-basis $\mathcal{B}$ formed by the sets

$$\{\phi \in \mathcal{F}(X) : \ a < \phi(x) < b\} \qquad x \in X, \ a, b \in \mathbb{R}. \tag{10}$$

Note that this topology is Hausdorff: suppose that f and g are two elements of $\mathcal{F}(X)$ that are not equal: there exists some $x \in X$ such that $f(x) \neq g(x)$; let $\epsilon = |f(x) - g(x)|$. The open sets

$$\{\phi \in \mathcal{F}(X) : \ f(x) - \epsilon/2 < \phi(x) < f(x) + \epsilon/2\}$$

(which contains f) and

$$\{\phi \in \mathcal{F}(X) : \ g(x) - \epsilon/2 < \phi(x) < g(x) - \epsilon/2\}$$

(which contains g) are disjoint.

Lemma 9.2. *If $(f_n) \in \mathcal{F}(X)$ then $f_n \to f$ in the topology of pointwise convergence if and only if $f_n(x) \to f(x)$ for every $x \in X$.*

Proof. Suppose that $f_n \to f$ in $\mathcal{T}_p$. Choose any $x \in X$ and $\epsilon > 0$; then

$$U := \{\phi \in \mathcal{F}(X) : \ f(x) - \epsilon < \phi(x) < f(x) + \epsilon\}$$

is an open set that contains f, so $f_n \in U$ for all $n \geq N$, i.e.

$$|f_n(x) - f(x)| < \epsilon \qquad \text{for all } n \geq N,$$

so $f_n(x) \to f(x)$.

To prove the reverse implication, suppose that $f_n \to f$ pointwise, and take any open set $U \in \mathcal{T}_p$ containing f. Then U is the union of finite intersections of elements of $\mathcal{B}$; choose one of these finite intersections that contains f,

$$V = \bigcap_{j=1}^{n} \{\phi \in \mathcal{F}(X) : \ a_j < \phi(x_j) < b_j\}.$$

Now, since for each $j = 1, \dots, n$ we have

$$f \in \{\phi \in \mathcal{F}(X) : \ a_j < \phi(x_j) < b_j\},$$

it follows that $a_j < f(x_j) < b_j$. Since $f_n \to f$ pointwise, we can find N_j such that $a_j < f_n(x_j) < b_j$ for all $n \geq N_j$. Now take $N = \max_{j=1,\dots,n} N_j$, and then $f_n \in V \subset U$ for all $n \geq N$. □

Now we show that this topology is not metrisable. To do this, we note that every closed set in a metrisable space can be written as the intersection of countably many open sets: if the topology comes from a metric d, then we have

$$S = \bigcap_{k=1}^{\infty} \left\{ x : \ d(x, S) < \frac{1}{k} \right\}.$$

[An intersection of countably many open sets is called a G_δ set.]

Theorem 9.3. *The space $\mathcal{F}[0,1]$ with the topology of pointwise convergence is not metrisable.*

Proof. First note that $\{0\}$, i.e. the function $f : [0,1] \to \mathbb{R}$ with $f(x) = 0$ for every $x \in [0,1]$ is closed: take any $f \in \mathcal{F}[0,1] \setminus \{0\}$. Then f has at least one point x with $f(x) \neq 0$; set $\epsilon = |f(x)|$ this is contained in

$$\{\phi \in \mathcal{F}(X) : f(x) - \epsilon < \phi(x) < f(x) + \epsilon\},$$

which is a subset of $\mathcal{F} \setminus \{0\}$.

If the space was metrisable, the set $\{0\}$ would be the countable intersection of open sets, i.e. we could find open sets $G_k \in \mathcal{F}[0,1]$ such that

$$\{0\} = \bigcap_{k=1}^{\infty} G_k.$$

For each k we know that G_k is the union of finite intersections of sets of the form equation (10), at least one of which must contain $\{0\}$. So taking one of these intersections we have

$$\{0\} \in \bigcap_{j=1}^{m} \{\phi : \ a_j < \phi(x_j) < b_j\} \subset G_k.$$

Since we know that $\{0\}$ is in each of the sets in the intersection, we can find an $\epsilon_k > 0$ such that $a_j \leq -\epsilon_k < 0 < \epsilon_k \leq b_j$ for each j [take $\epsilon_k = \min_{j=1}^{m}(-a_j, b_j)$], and then we have

$$\{0\} \in \bigcap_{j=1}^{m} \{\phi : \ -\epsilon_k < \phi(x_j) < \epsilon_k\} \subset G_k.$$

If we let A_k be the finite set $\{x_1, \ldots, x_j\}$ then we can write this as

$$\{0\} \in \{\phi : \ -\epsilon_k < \phi(x) < \epsilon_k, \ x \in A_k\} \subset G_k.$$

But now the function $g : [0,1] \to \mathbb{R}$ defined by setting

$$g(x) = \begin{cases} 0 & x \in A_k, \text{ for some } k \\ 1 & \text{otherwise} \end{cases}$$

is a non-zero element of

$$\bigcap_k \{\phi : \ -\epsilon_k < \phi(x) < \epsilon_k, \ x \in A_k\} \subset \bigcap_k G_k.$$ □

9.2 Product spaces

This is a quick note on products of arbitrary collections of sets and how they are defined as a set of functions, which is initially quite confusing. It does not address putting a topology on the product, just how the product is defined.

Let's start with the product $X_1 \times X_2$ of two sets X_1 and X_2. We are familiar with this being defined as a set of ordered pairs:

$$X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1, \ x_2 \in X_2\}.$$

Similarly, if we have n sets X_1,..., X_n, we define the product

$$X_1 \times \cdots \times X_n = \prod_{i=1}^{n} X_i$$

to be a set of ordered n-tuples:

$$X_1 \times \cdots \times X_n = \prod_{i=1}^{n} X_i = \{(x_1, \ldots, x_n) : x_i \in X_i \ \forall i \in \{1, \ldots, n\}\}.$$

(Note that $\prod_{i=1}^{n} X_i$ is just notation for $X_1 \times \cdots \times X_n$, in the same way as we write $\sum_{i=1}^{n} a_n$ for $a_1 + \cdots + a_n$.)

Now suppose we have a collection of sets $X_1, X_2, X_3, \ldots$, indexed by the natural numbers $\mathbb{N}$. We can define the product of these sets to be a set of sequences:

$$\begin{aligned} \prod_{i=1}^{\infty} X_i = \prod_{i \in \mathbb{N}} X_i &= \{(x_1, x_2, x_3, \ldots) : x_i \in X_i \ \forall i \in \mathbb{N}\} \\ &= \{(x_i)_{i=1}^{\infty} : x_i \in X_i \ \forall i \in \mathbb{N}\}. \end{aligned}$$

Note that the expression $(x_i)_{i=1}^{\infty}$ is just a shorthand way of writing $(x_1, x_2, x_3, \ldots)$.

If we have a collection of sets indexed by a countably infinite set, i.e., sets X_i for $i \in C$, where C is countably infinite, we can use the fact that C is in bijection with $\mathbb{N}$ to order the elements. We can then again define the product $\prod_{i \in C} X_i$ to be a set of sequences.

Now suppose we have a collection of sets X_i, $i \in \Lambda$, where Λ is an arbitrary uncountable set. How can we make sense of

$$\prod_{i \in \Lambda} X_i \ ?$$

We might be tempted to write the elements formally as $(x_i)_{i \in \Lambda}$ with $x_i \in X_i$ but what does this notation mean?

To make sense of such products over arbitrary collections of sets, we have to take a step back and reinterpret our earlier examples.

Consider again the product $X_1 \times X_2$. We claim that a point $(x_1, x_2) \in X_1 \times X_2$ specifies a function from $\{1, 2\}$ to $X_1 \cup X_2$. We'll call this function $x : \{1, 2\} \to X_1 \cup X_2$ and it is defined by $x(1) = x_1$ and $x(2) = y_1$. If we take another point $(y_1, y_1) \in X_1, X_2$, we get another function, which we'll call $y : \{1, 2\} \to X_1 \cup X_2$, defined by $y(1) = y_1$ and $y(2) = y_2$. However, these functions must all satisfy a restriction: the value they take at 1 must lie in X_1 and the value they take at 2 must lie in X_2. So each $(x_1, x_2) \in X_1 \times X_2$ gives a function $x : \{1, 2\} \to X_1 \cup X_2$ such that $x(i) \in X_i$ for all $i \in \{1, 2\}$.

Now let's go from the other direction. Suppose we have a function $x : \{1, 2\} \to X_1 \cup X_2$ such that $x(i) \in X_i$ for all $i \in \{1, 2\}$. This determines a point $(x_1, x_2) \in X_1 \times X_2$ by setting $(x_1, x_2) = (x(1), x(2))$.

So we have a natural bijection between $X_1 \times X_2$ and the set of functions

$$\{x : \{1, 2\} \to X_1 \cup X_2 : x(i) \in X_i \ \forall i \in \{1, 2\}\}.$$

Similarly, given sets $X_1, \ldots, X_n$, there is a natural bijection between $X_1 \times \cdots \times X_n$ and the set of functions

$$\{x : \{1, \ldots, n\} \to X_1 \cup \cdots \cup X_n : x(i) \in X_i \ \forall i \in \{1, \ldots, n\}\}.$$

Generalising a bit more, for sets $X_1, X_2, X_3, \ldots$, indexed by $\mathbb{N}$, there is a natural bijection between $\prod_{i \in \mathbb{N}} X_i$ and the set of functions

$$\left\{ x : \mathbb{N} \to \bigcup_{i \in \mathbb{N}} X_i : x(i) \in X_i \ \forall i \in \mathbb{N} \right\}.$$

The bijection is given by mapping $(x_i)_{i=1}^{\infty}$ to the function $x : \mathbb{N} \to \bigcup_{i \in \mathbb{N}} X_i$ defined by $x(i) = x_i$ for all $i \in \mathbb{N}$.

Now let us return to an arbitrary collection of sets X_i, $i \in \Lambda$. Given the discussion above, it is natural to define the product set

$$\prod_{i\in\Lambda} X_i$$

to be the set of functions

$$\left\{ x : \Lambda \to \bigcup_{i\in\Lambda} X_i \,:\, x(i) \in X_i \;\forall i \in \Lambda \right\}.$$

We can write an element of this set as $(x_i)_{i\in\Lambda}$ and this means the function $x : \Lambda \to \bigcup_{i\in\Lambda} X_i$ defined by $x(i) = x_i$ for all $i \in \Lambda$.

9.3 Product topology and box topology

This note gives a bit more detail about the definitions of the product topology and the box topology on the product of an infinite collection of topological spaces. We work out both topologies in the simplest example where they are different. You don't really need to worry about the box topology but seeing it helps one to understand why the product topology (for an infinite collection of sets) is defined as it is.

Let $(T_1, \mathcal{T}_1), \ldots, (T_n, \mathcal{T}_n)$ be a finite collection of topological spaces. We defined the *product topology* on the product space

$$T_1 \times \cdots \times T_n = \{(x_1, \ldots, x_n) \,:\, x_i \in T_i \;\forall i \in \{1, \ldots, n\}\}.$$

to be the topology with basis

$$\mathcal{B}_0 = \{U_1 \times \cdots \times U_n \,:\, U_i \in \mathcal{T}_i \;\forall i \in \{1, \ldots, n\}\}.$$

Note:

1. For $U_1 \times \cdots \times U_n \in \mathcal{B}_0$ and $V_1 \times \cdots \times V_n \in \mathcal{B}_0$,

   $$(U_1 \times \cdots \times U_n) \cap (V_1 \times \cdots \times V_n) = (U_1 \cap V_1) \times \cdots \times (U_n \cap V_n) \in \mathcal{B}_0,$$

   since each $U_i \cap V_i \in \mathcal{T}_i$. Therefore, $\mathcal{B}_0$ is the basis for a topology.

2. The topology defined by the basis is unique.

Now we want to define a topology on the product of an infinite collection of sets. In fact, there are two topologies we can define but one of these is better than the other.

We can define these topologies on the product of an arbitrary collection of spaces

$$\prod_{i\in\Lambda} T_i = \left\{ x : \Lambda \to \bigcup_{i\in\Lambda} T_i \,:\, x(i) \in T_i \;\forall i \in \Lambda \right\}$$

but the ideas will show up clearly if we just consider a countable collection of spaces, indexed by $\mathbb{N}$.

So suppose we have topological spaces $(T_1, \mathcal{T}_1), (T_2, \mathcal{T}_2), (T_3, \mathcal{T}_3), \ldots$ and we want to define a topology on

$$T := \prod_{i=1}^{\infty} T_i = \{(x_i)_{i=1}^{\infty} : x_i \in T_i \ \forall i \in T_i\}.$$

The most obvious (and bad) way to define a topology on T is to take the topology with basis

$$\mathcal{B}_{\text{box}} := \left\{ \prod_{i=1}^{\infty} U_i : U_i \in \mathcal{T}_i \ \forall i \in \mathbb{N} \right\}.$$

Here,

$$\prod_{i=1}^{\infty} U_i = \{(x_i)_{i=1}^{\infty} : x_i \in U_i \ \forall i \in T_i\}.$$

As above, you can check the the intersection of two sets in $\mathcal{B}_{\text{box}}$ is still in $\mathcal{B}_{\text{box}}$, so $\mathcal{B}_{\text{box}}$ is the basis for a unique topology. This is called the *box topology* $\mathcal{T}_{\text{box}}$.

A less obvious approach is to use the projective topology with respect to the projection maps onto the factors T_i. For each $j \in \mathbb{N}$, define the projection map $\pi_j : T \to T_j$ by $\pi_i((x)_{i=1}^{\infty}) = x_j$. The *product topology* $\mathcal{T}$ is defined to be the smallest (coarsest) topology that makes all of the maps $\pi_j : T \to T_j$, $j \in \mathbb{N}$, continuous.

Let's see what sort of open sets we get from the continuity of the maps π_j. Start with $j = 1$. Since π_1 is continuous, for every $U_1 \in \mathcal{T}_1$,

$$\pi_1^{-1}(U_1) = U_1 \times T_2 \times T_3 \times \cdots = U_1 \times \prod_{i=2}^{\infty} T_i \in \mathcal{T}.$$

Similarly, since π_2 is continuous, for every $U_2 \in \mathcal{T}_2$,

$$\pi_2^{-1}(U_2) = T_1 \times U_2 \times T_3 \times \cdots = T_1 \times U_2 \times \prod_{i=3}^{\infty} T_i \in \mathcal{T}.$$

And so on: for each $j \in \mathbb{N}$ and every $U_j \in \mathcal{T}_j$,

$$\pi_j^{-1}(U_j) = \prod_{i=1}^{j-1} T_i \times U_j \times \prod_{i=j+1}^{\infty} T_i \in \mathcal{T}.$$

Collecting these sets together, we get

$$\mathcal{S} = \bigcup_{j=1}^{\infty} \left\{ \prod_{i=1}^{\infty} U_i : U_i \in \mathcal{T}_i \ \forall i \in \mathbb{N} \text{ and } U_i = T_i \text{ for } i \neq j \right\}.$$

The product topology is smallest topology which contains these sets, which is the same as saying that it is the unique topology for which $\mathcal{S}$ is a sub-basis.

You should check that the intersection of two sets in $\mathcal{S}$ is, in general, not a union of sets in $\mathcal{S}$ and so $\mathcal{S}$ is not a basis. However, we can get a basis by considering finite intersections of sets in $\mathcal{S}$. Such an intersection will have the form

$$\prod_{i=1}^{\infty} U_i^{(1)} \cap \prod_{i=1}^{\infty} U_i^{(2)} \cap \cdots \cap \prod_{i=1}^{\infty} U_i^{(n)} = \prod_{i=1}^{\infty} (U_i^{(1)} \cap U_i^{(2)} \cap \cdots \cap U_i^{(n)}).$$

We know that for each $k = 1, \ldots, n$ there is at most one value of i for which $U_i^{(k)} \neq T_i$, so there are at most finitely many values of i (in fact, at most n values) for which $U_i^{(1)} \cap U_i^{(2)} \cap \cdots \cap U_i^{(n)} \neq T_i$. So a basis for the product topology is given by

$$\mathcal{B} = \left\{ \prod_{i=1}^{\infty} U_i : U_i \in \mathcal{T}_i \; \forall i \in \mathbb{N} \text{ and } U_i = T_i \text{ for all except finitely many values of } i \right\}.$$

You should note that, for the product of a finite collection of sets, the product topology defined in the last paragraph is equal to the box topology (and both are equal to the product topology defined at the start). However, for an infinite collection of sets, the product topology and the box topology are different.

We claim that the product topology is a better, i.e. more useful, topology than the box topology. Why? One answer is given by the following:

Tychonov's Theorem (Theorem 6.14). If $(T_i, \mathcal{T}_i)$, $i \in \Lambda$, are compact topological spaces then $T = \prod_{i \in \Lambda} T_i$ with the product topology is also compact.

This is not true for the box topology (if Λ is infinite) and we'll see a simple example. In fact, it is instructive to look at this example to see why the product topology is a better topology to use.

We'll again take a countable collection of sets T_i, $i \in \mathbb{N}$, but take all the T_i to be the same set $\{0, 1\}$ with the discrete topology. So

$$\prod_{i=1}^{\infty} T_i = \{(x_i)_{i=1}^{\infty} : x_i \in \{0, 1\} \; \forall i \in \mathbb{N}\},$$

i.e. the set of all sequences of 0's and 1's. We looked at this space before, when we called it X_∞, so we'll continue with that notation. Recall that we can define a metric d_∞ on X_∞ by

$$d_\infty((x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty}) = \sum_{i=1}^{\infty} \frac{1 - \delta_{x_i y_i}}{2^i}$$

(where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$). In fact, it is a bit nicer to work with the following metric d:

$$d((x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty) = \begin{cases} 0 & \text{if } (x_i)_{i=1}^\infty = (y_i)_{i=1}^\infty \\ \frac{1}{2^{n-1}} & \text{if } (x_i)_{i=1}^\infty \neq (y_i)_{i=1}^\infty \text{ and } n = \min\{i \in \mathbb{N} : x_i \neq y_i\}. \end{cases}$$

So, for example, if $x_1 \neq y_1$ then $d((x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty) = 1$. For every $(x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty \in X_\infty$, we have

$$\frac{1}{2} d((x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty) \leq d_\infty((x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty) \leq d((x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty),$$

so d and d_∞ are Lipschitz equivalent and hence topologically equivalent. Notice that d has the following two properties:

1. $d((x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty) \leq 2^{-n}$ if and only if $d((x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty) < 2^{-(n-1)}$;
2. if $B(x, r)$ is an open ball in the metric d then $B(x, r) = B(x, 2^{-n})$, for some $n \geq 0$.

Let's think about the box topology on X_∞. Let $x = (x_i)_{i=1}^\infty$. Since each $T_i = \{0, 1\}$ has the discrete topology, $\{x_i\}$ is an open set in T_i. Hence

$$\prod_{i=1}^\infty \{x_i\} \in \mathcal{B}_{\text{box}} \subset \mathcal{T}_{\text{box}}.$$

But

$$\prod_{i=1}^\infty \{x_i\} = \{x\},$$

so sets containing single points are in $\mathcal{T}_{\text{box}}$. Taking unions, we see that every subset of X_∞ is in $\mathcal{T}_{\text{box}}$, i.e. the box topology is the discrete topology.

We can now see that $(X_\infty, \mathcal{T}_{\text{box}})$ is not compact: $\mathcal{U} = \{\{x\} : x \in X_\infty\}$ is an open cover for $(X_\infty, \mathcal{T}_{\text{box}})$ but there can be no finite subcover as X_∞ is an infinite set.

We also see that a sequence $x^{(n)} = (x_i^{(n)})_{i=1}^\infty$ converges to $x = (x_i)_{i=1}^\infty$ in the box topology if and only if it is eventually equal to x, i.e. that there exists $N \geq 1$ such that $x^{(n)} = x$ for all $n \geq N$. (This is because $\{x\}$ is an open neighbourhood of x.)

Now let's contrast this with the product topology on X_∞. We claim that the product topology $\mathcal{T}$ is equal to the topology $\mathcal{T}_d$ induced by the metric d. We will now justify this claim.

For $x = (x_i)_{i=1}^\infty$ and $m \geq 1$, define sets

$$C_m(x) := \{x_1\} \times \cdots \times \{x_m\} \times \prod_{i=m+1}^\infty T_i.$$

These are open sets in the product topology which contain x. Furthermore, we see that

$$\begin{aligned} C_m(x) &= \{y = (y_i)_{i=1}^\infty : x_i = y_i \ \forall i \in \{1, \ldots, m\}\} \\ &= \{y = (y_i)_{i=1}^\infty : d(x, y) \leq 2^{-m}\} \\ &= \{y = (y_i)_{i=1}^\infty : d(x, y) < 2^{-(m-1)}\} \text{ (using property (1) above)} \\ &= B(x, 2^{-(m-1)}), \end{aligned}$$

the open ball in the metric d centred at x with radius $2^{-(m-1)}$. So (using property (2) above) every open ball for the metric d is open in the product topology. Since the open balls for the metric d are a basis for $\mathcal{T}_d$, we have $\mathcal{T}_d \subset \mathcal{T}$.

Next we prove the reverse inclusion. Let $V \in \mathcal{T}$ be non-empty and let $x \in V$. Since V is a union of sets in the basis $\mathcal{B}$ for the product topology, we can find a set

$$H_x = \prod_{i=1}^\infty U_i \in \mathcal{B}$$

such that $x \in H_x \subset V$. Let $m(x)$ be the largest value of i for which $U_i \neq T_i = \{0, 1\}$. Then

$$x \in C_{m(x)}(x) \subset H_x \subset V.$$

Hence

$$V = \bigcup_{x \in V} C_{m(x)}(x) = \bigcup_{x \in V} B(x, 2^{-(m(x)-1)}),$$

so V is open in the topology induced by the metric. Therefore $\mathcal{T} \subset \mathcal{T}_d$. This completes the proof of the claim.

Finally, let us see what convergence of sequences looks like in X_∞ with the product topology. We see $x^{(n)} = (x_i^{(n)})_{i=1}^\infty$ converges to $x = (x_i)_{i=1}^\infty$ if and only if for every $m \geq 1$ there exists $N \geq 1$ such that

$$n \geq N \quad \implies \quad x_i^{(n)} = x_i \ \forall i \in \{1, \ldots, m\}.$$

So, for example, the sequence (of sequences)

$$\begin{aligned} x^{(1)} &= (1, 1, 1, 1, \ldots), \\ x^{(2)} &= (0, 1, 1, 1, \ldots), \\ x^{(3)} &= (0, 0, 1, 1, \ldots), \\ &\vdots \\ x^{(n)} &= (\underbrace{0, 0, \ldots, 0}_{n-1 \text{ terms}} 1, 1, 1, \ldots) \\ &\vdots \end{aligned}$$

(i.e. $x^{(n)}$ has $n-1$ 0's followed by all 1's) converges to

$$x = (0, 0, 0, 0, \ldots).$$

9.4 Completeness in compact metric spaces

Definition 9.4. A metric space (X, d) is *totally bounded* (also called *precompact*) if for every $\epsilon > 0$ there is a finite ϵ-net in X, i.e. X can be covered by a finite collection of balls of radius ϵ:

$$X \subset \bigcup_{j=1}^{n} B(x_j, \epsilon).$$

Note that any totally bounded set is bounded. (We have $X \subset \bigcup_{j=1}^{n} B(x_j, 1)$, so for every $x \in X$ we have $d(x, x_1) < r := 1 + \max_j d(x_1, x_j)$.) The converse is not true - consider any infinite set with the discrete metric.

Lemma 9.5. *A subspace Y of a metric space (X, d) is totally bounded if and only if for every $\epsilon > 0$ there is a finite collection $x_1, \ldots, x_n \in X$ such that*

$$Y \subset \bigcup_{j=1}^{n} B(x_j, \epsilon).$$

Proof. Only one direction needs proof (if $x_j \in Y$ then $x_j \in X$).

Given $\epsilon > 0$ find a collection $\{x_1, \ldots, x_n\}$ such that

$$Y \subset \bigcup_{j=1}^{n} B(x_j, \epsilon/2).$$

We can assume that $Y \cap B(x_j, \epsilon/2) \neq \varnothing$ for each j; otherwise we would just remove the ball centred at x_j from the cover.

Now for each j choose one point $y_j \in Y \cap B(x_j, \epsilon/2)$; then

$$B(x_j, \epsilon/2) \subset B(y_j, \epsilon),$$

and so

$$Y \subset \bigcup_{j=1}^{n} B(y_j, \epsilon)$$

as required. □

Lemma 9.6. *A subspace of a totally bounded metric space is totally bounded.*

Proof. If $Y \subset X$ and X is totally bounded then for every $\epsilon > 0$ there are $\{x_1, \dots, x_n\} \in X$ such that

$$Y \subset X \subset \bigcup_{j=1}^{n} B(x_j, \epsilon),$$

so Y is totally bounded using Lemma 9.5. □

Lemma 9.7. *If a subspace Y of a metric space X is totally bounded then so is $\overline{Y}$.*

Proof. Given $\epsilon > 0$ let $\{x_1, \dots, x_n\}$ be an $\epsilon/2$-net for Y. Then this is an ϵ-net for $\overline{Y}$, since given any $y \in \overline{Y}$ there exists $x \in Y$ with $d(x, y) < \epsilon/2$ and x_i such that $d(x, x_i) < \epsilon/2$, so $d(y, x_i) < \epsilon$. □

Proposition 9.8. *Any sequence in a totally bounded metric space (X, d) has a Cauchy subsequence.*

Proof. Take a sequence $(x_n) \in X$.

Since X is totally bounded, it can be covered by finitely many 1/2-balls. So there is at least one ball $B(y_1, 1/2)$ containing infinitely many elements of the sequence (x_n). All elements of this subsequence are within 1 of each other.

Choose n_1 such that $x_{n_1} \in B(y_1, 1/2)$, and let

$$X_1 = \{x_j : \ j > n_1,\ x_j \in B(y_1, 1/2)\}.$$

Since X can be covered by finitely many 1/4 balls, so there is one, $B(y_2, 1/4)$, say that contains infinitely many of the points in X_1 (and all of these points are within 1/2 of each other). Choose n_2 such that $x_{n_2} \in B(y_2, 1/4)$ and let

$$X_2 = \{x_j : \ j > n_2,\ x_j \in B(y_2, 1/4)\}.$$

Continuing in this way we obtain a subsequence (x_{n_j}) of (x_n) that is Cauchy, since $x_{n_i} \in B(y_j, 2^{-j})$ for all $i \geq j$. □

Theorem 9.9. *A subspace Y of a complete metric space (X, d) is compact if and only if it is closed and totally bounded.*

Proof. If Y is compact then Y is closed by Lemma 6.7 and totally bounded since the open cover $\{B(x, \epsilon) : x \in Y\}$ has a finite subcover.

Conversely, if Y is totally bounded then any sequence in Y has a Cauchy subsequence. Since X is complete this subsequence converges; since Y is closed the limit of the sequence lies in Y. So Y is sequentially compact; since (Y, d) is a metric space this implies that Y is compact. □

Theorem 9.10. *A subspace Y of a complete metric space is totally bounded iff its closure is compact.*

Proof. If Y is totally bounded then $\overline{Y}$ is totally bounded by Lemma 9.7 and so compact by the previous theorem.

If $\overline{Y}$ is compact then it is totally bounded by the previous theorem) and so Y is totally bounded by Lemma 9.6. □

9.5 The general Arzelà–Ascoli theorem

The Arzelà–Ascoli Theorem gives a characterisation of compact subsets of $C(X)$, when X is a compact metric space.

Definition 9.11. A subset S of $C(X)$ is

- *equicontinuous at* x if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$y \in B(x,\delta) \qquad \Rightarrow \qquad |f(y) - f(x)| < \epsilon \qquad \text{for every } f \in S;$$

- *equicontinuous* if it is equicontinuous at every $x \in X$;
- *uniformly equicontinuous* if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d(y,x) < \delta \qquad \Rightarrow \qquad |f(y) - f(x)| < \epsilon \qquad \text{for every } f \in S.$$

Lemma 9.12. *If X is compact then $S \subset C(X)$ is equicontinuous if and only if it is uniformly equicontinuous.*

Proof. This is an exercise on Problem Sheet 6. □

Theorem 9.13. *Let X be a compact metric space. A subset A of $C(X)$ is totally bounded if and only if it is bounded and equicontinuous.*

Proof. If A is totally bounded then it is bounded. Since A is totally bounded, for any $\epsilon > 0$ there exist $\{f_1, \dots, f_n\}$ such that for every $f \in A$ there is an i with

$$\|f - f_i\|_\infty < \epsilon.$$

Since each of the f_i are uniformly continuous there exists $\delta > 0$ such that for $i = 1, \dots, n$

$$d(x,y) < \delta \qquad \Rightarrow \qquad |f_i(x) - f_i(y)| < \epsilon/3.$$

Then for any $f \in A$ choose j such that $\|f_j - f\|_\infty < \epsilon/3$; it follows that if $d(x,y) < \delta$ we have

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| \\ &\le \|f - f_j\|_\infty + |f_j(x) + f_j(y)|_{\|} f_j - f\|_\infty < \epsilon, \end{aligned}$$

so A is uniformly equicontinuous (and hence equicontinuous).

Now, assuming that A is bounded and equicontinuous, given $\epsilon > 0$ we want to find a finite ϵ-net in A. For every $x \in X$ use the fact that A is equicontinuous to find $\delta(x) > 0$ such that

$$y \in B(x, \delta(x)) \qquad \Rightarrow \qquad |f(y) - f(x)| < \epsilon/3 \qquad \text{for every } f \in A.$$

Since X is compact there is a finite set $\{x_1, \ldots, x_n\}$ such that

$$X \subset \bigcup_{i=1}^{n} B(x_i, \delta(x_i)).$$

We now make a collection F of elements of A that form a finite ϵ-net. For any $\{q_1, \ldots, q_n\}$ with $q_i \in \mathbb{Z}$ for which there exists a $g \in A$ with

$$g(x_i) \in [q_i\epsilon/3, (q_i + 1)\epsilon/3]$$

we choose one such g and add it to F. Since A is bounded there are only finitely many such choices of $\{q_1, \ldots, q_n\}$, so there are only finitely many functions in F.

Now given any $f \in A$ for each i there are q_i such that

$$f(x_i) \in [q_i\epsilon/3, (q_i + 1)\epsilon/3],$$

and so there is a $g \in F$ such that

$$g(x_i) \in [q_i\epsilon/3, (q_i + 1)\epsilon/3],$$

which implies that $|f(x_i) - g(x_i)| < \epsilon/3$ for each $i = 1, \ldots, n$.

Now for each $x \in X$ we can find j such that $x \in B(x_j, \delta(x_j))$, and then

$$\begin{aligned} |f(x) - g(x)| &\leq |f(x) - f(x_j)| + |f(x_j) - g(x_j)| + |g(x_j) - g(x)| \\ &\leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon, \end{aligned}$$

from which it follows that $\|f - g\|_\infty < \epsilon$, i.e. A is totally bounded. □

Corollary 9.14 (Arzelà–Ascoli Theorem, general form)**.** *Let X be a compact metric space. A subset A of $C(X)$ is compact if and only if it is closed, bounded, and equicontinuous.*

One application is the following result.

Theorem 9.15. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Then there exists $\delta > 0$ such that the differential equation*

$$\dot{x} = f(x), \qquad x(0) = x_0$$

has at least one solution for $t \in (-\delta, \delta)$.